Modeling the Precession and Nutation of a Gyroscope

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AAPT 1998 Summer Meeting
University of Nebraska
Lincoln, Nebraska

Equations of Motion of the Gyroscope

$$\omega_{x'} = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi$$

$$\omega_{y'} = \dot{\phi}\sin\theta\cos\psi + \dot{\theta}\sin\psi$$

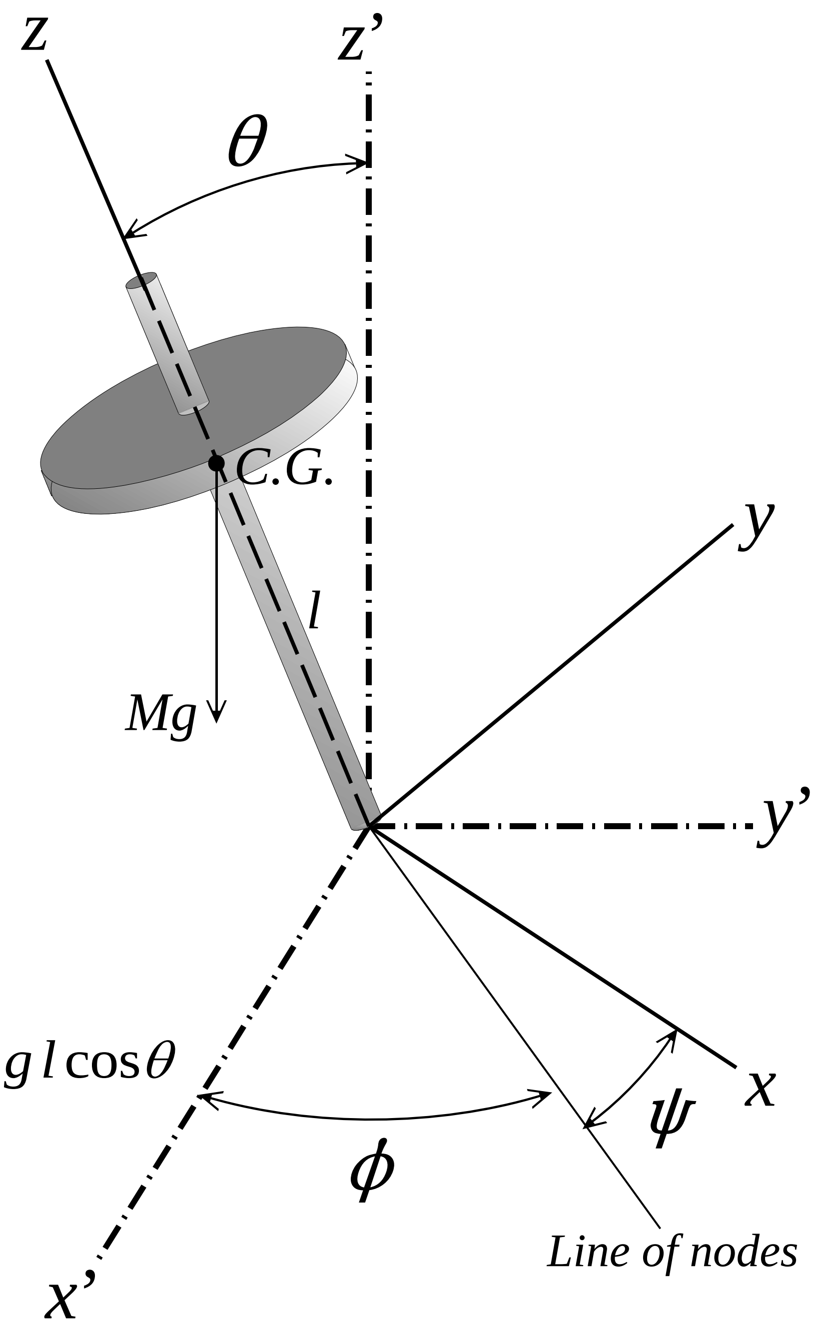
$$\omega_{z'} = \dot{\phi}\cos\theta + \dot{\psi}$$

$$T = \frac{1}{2} I_1 (\omega_x^2 + \omega_y^2) + \frac{1}{2} I_3 \omega_z^2$$

$$T = \frac{I_1}{2}(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + \frac{I_3}{2}(\dot{\psi} + \dot{\phi}\cos\theta)^2$$

$$V = M\,g\,l\cos\theta$$

$$L = \frac{I_1}{2}(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + \frac{I_3}{2}(\dot{\psi} + \dot{\phi}\cos\theta)^2 - M\,g\,l\cos\theta$$



Equations of Motion

$$\frac{d}{dt} p_\psi = Q_\psi = I_1 a_1$$

so $$p_\psi = \frac{\partial L}{\partial \dot\psi} = I_3(\dot\psi + \dot\phi \cos\theta) = I_1 a_1 \mathrm{t} + I_1 \mathrm{a_o} = I_1 a$$

$$\frac{d}{dt} p_\phi = Q_\phi = I_1 b_1$$

so $$p_\phi = \frac{\partial L}{\partial \dot\phi} = (I_1 \sin^2\theta + I_3 \cos^2\theta)\dot\phi + I_3 \dot\psi \cos\theta = I_1 b_1 \mathrm{t} + I_1 \mathrm{b_o} = I_1 b$$

$$E = T + V = \frac{I_1}{2}(\dot\theta^2 + \dot\phi^2 \sin^2\theta) + \frac{I_1^2 a^2}{2 I_3} + M g l \cos\theta$$

Solving the p_ψ and p_ϕ for phidot and psidot gives

$$\dot\phi = \frac{b - a\cos\theta}{\sin^2\theta} \qquad \dot\psi = \frac{I_1 a}{I_3} - \frac{b - a\cos\theta}{\sin^2\theta}\cos\theta$$

Apparatus

- Gyroscope, Pasco ME-8960
- Rotary Motion Sensors (2), Pasco CI-6538
- Accessory Photogate, Pasco ME-9204B
- Computer Interfaces (2), Science Workshop 700, Pasco CI-6565A
- Computers (2), Dell 75MHz Pentiums

Experimental Procedure

- Calibrate rotary motion sensors
- Open and set up Science Workshop
- Set sampling rate to 50 Hz
- Begin collection of angular data on computer 1
- Take Ψdot data with photogate on computer 2
- Release gyroscope and record for about 20 s
- Take Ψdot data with photogate on computer 2

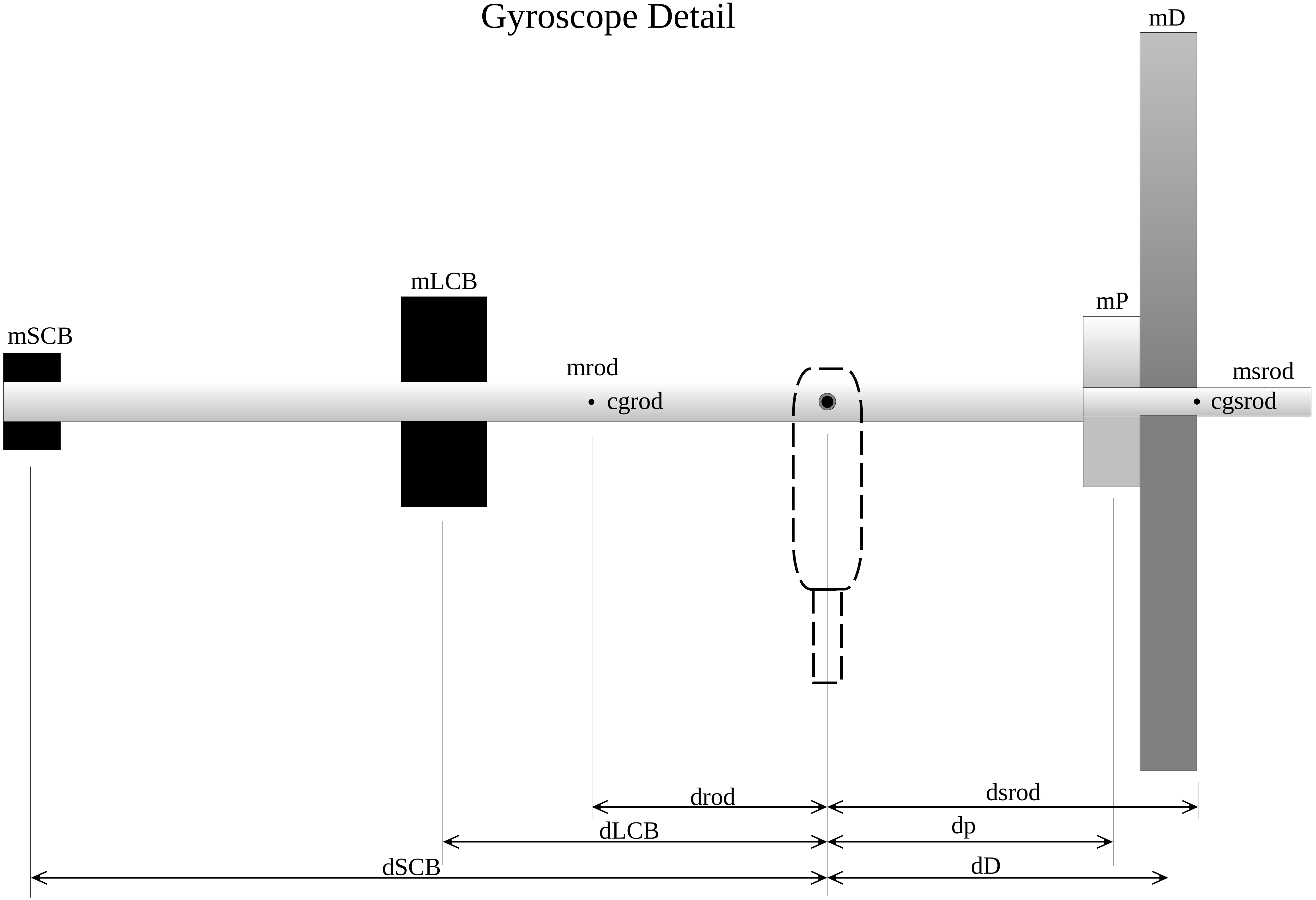
Calibration of Rotary ϕ Sensor

- Position gyroscope spin axis horizontally
- Begin recording data
- Keep gyroscope stationary for one second
- Rotate gyroscope spin axis horizontally one rotation
- Keep gyroscope stationary for one second
- Record mean reading at beginning and end
- Calculate calibration factor

Calibration of θ Rotary Sensor

- Position gyroscope spin axis horizontally
- Begin recording data
- Keep gyroscope stationary for one second
- Rotate gyroscope spin axis to $\theta = -35°$
- Keep gyroscope stationary for one second
- Rotate gyroscope spin axis to $\theta = 45°$
- Keep gyroscope stationary for one second
- Record mean reading at beginning and end
- Calculate calibration factor

Gyroscope Detail



Gyroscope Data

Component	Mass kg	Inner Radius m	Outer Radius m	CM Displace. m
Disk	1.735	.0060	.127	.1225
Large CB	.8974	.0065	.0350	-.1517
Small CB	.0470	.0066	.0225	-.2966
Pulley	.131	.0048	.0294	.1022

Axles	Mass	Radius	CM Displace.
Long rod	.14	.0064	-.108
Short rod	.016	.0048	.133

Calculation of Gyroscope Parameters

Total mass

$$M = mD + mLCB + mSCB + mp + mrod + msrod = 2.966\,\mathrm{kg}$$

Displacement to the center of mass

$$dcm = \frac{mD \cdot dD + mLCB \cdot dLCB + mSCB \cdot dSCB + mp \cdot dp + mrod \cdot drod + msrod \cdot dsrod}{M}$$

$$dcm = .0212\,\mathrm{m}$$

Moment of inertia about the spin axis, I_3

$$I_3 = \frac{1}{2} mD\left(RDo^2 + RDi^2\right) + \frac{1}{2} mp\left(Rpo^2 + Rpi^2\right) = .0140\,\mathrm{kg\ \ m^2}$$

Moment of Inertia about Axis Perpendicular to Spin Axis

Disk $$ID1 = \frac{1}{4} mD(RDo^2 + RDi^2) + mD \cdot dD^2 = .0325 \text{ kg m}^2$$

Large CB $$ILCB = \frac{1}{4} mLCB(RLCBo^2 + RLCBi^2) + mLCB \cdot dLCB^2 = .0209 \text{ kg m}^2$$

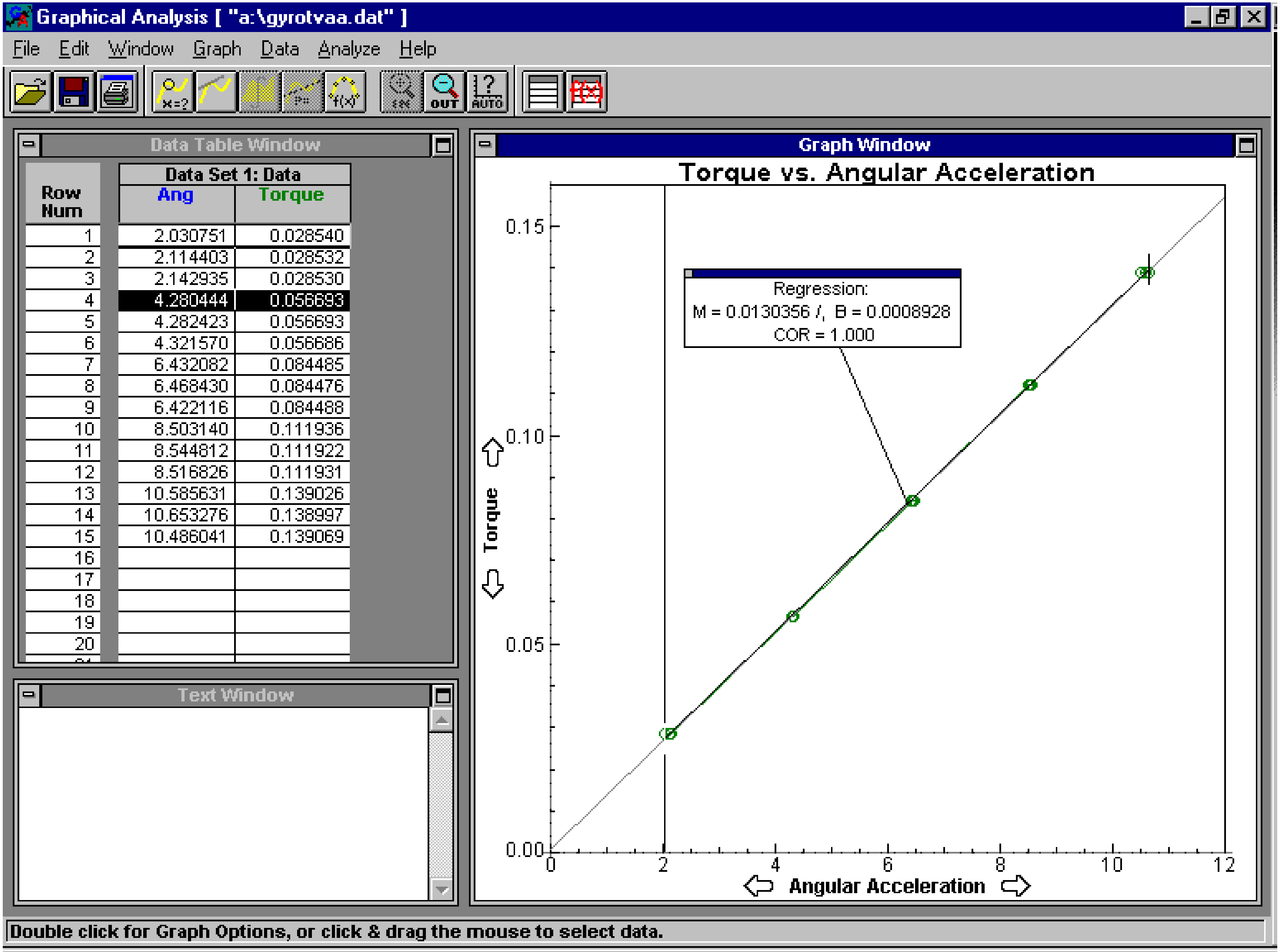
Small CB $$ISCB = \frac{1}{4} mSCB(RSCBo^2 + RSCBi^2) + mSCB \cdot dSCB^2 = .0041 \text{ kg m}^2$$

Pulley $$Ip = \frac{1}{4} mp(Rpo^2 + Rpi^2) + mp \cdot dp^2 = .0014 \text{ kg m}^2$$

Long Rod $$Irod = \frac{1}{12} mrod \cdot Lrod^2 + mrod \cdot drod^2 = .0016 \text{ kg m}^2$$

Short Rod $$Isrod = \frac{1}{12} msrod \cdot Lsrod^2 + msrod \cdot dsrod^2 = .0003 \text{ kg m}^2$$

Total $$I1 = ID1 + ILCB + ISCB + IP + Irod + Isrod = .0609 \text{ kg m}^2$$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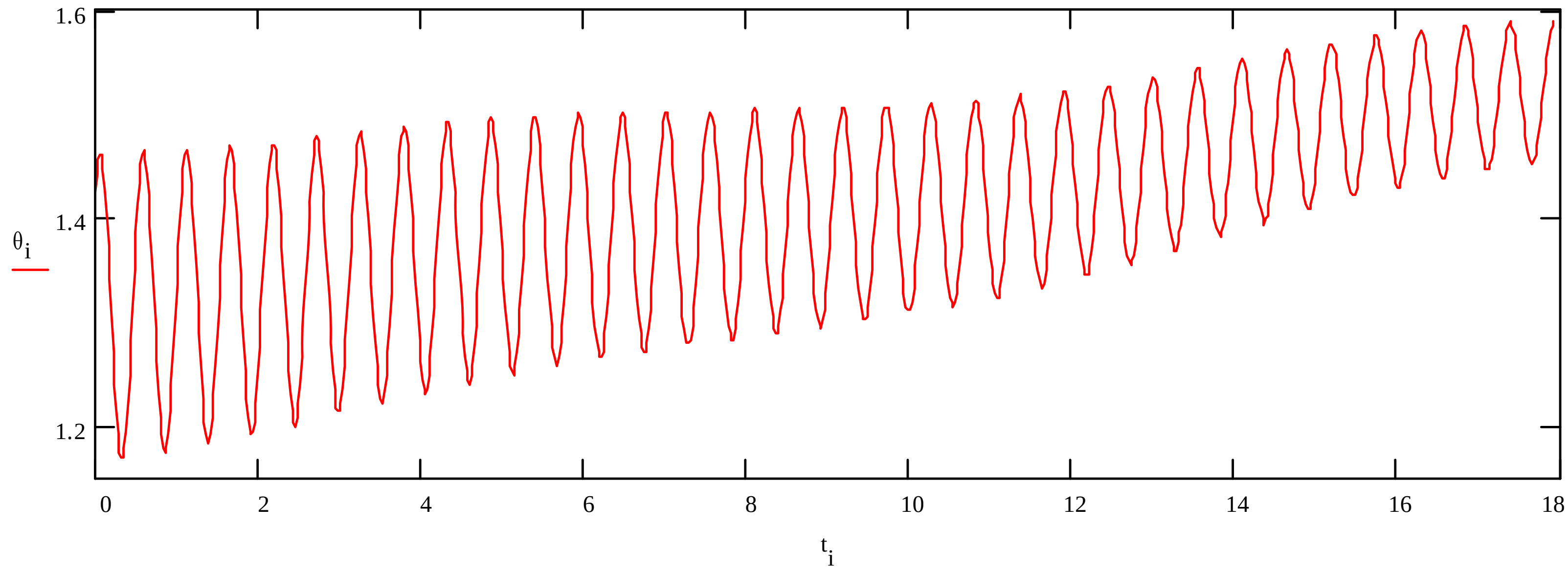
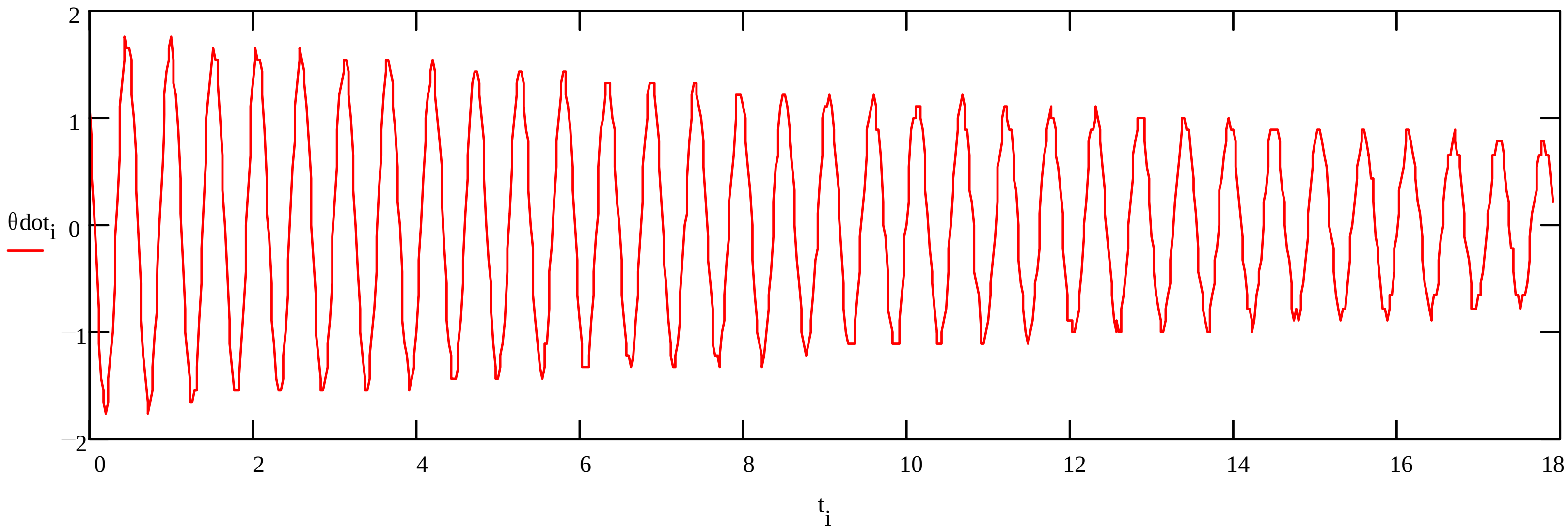
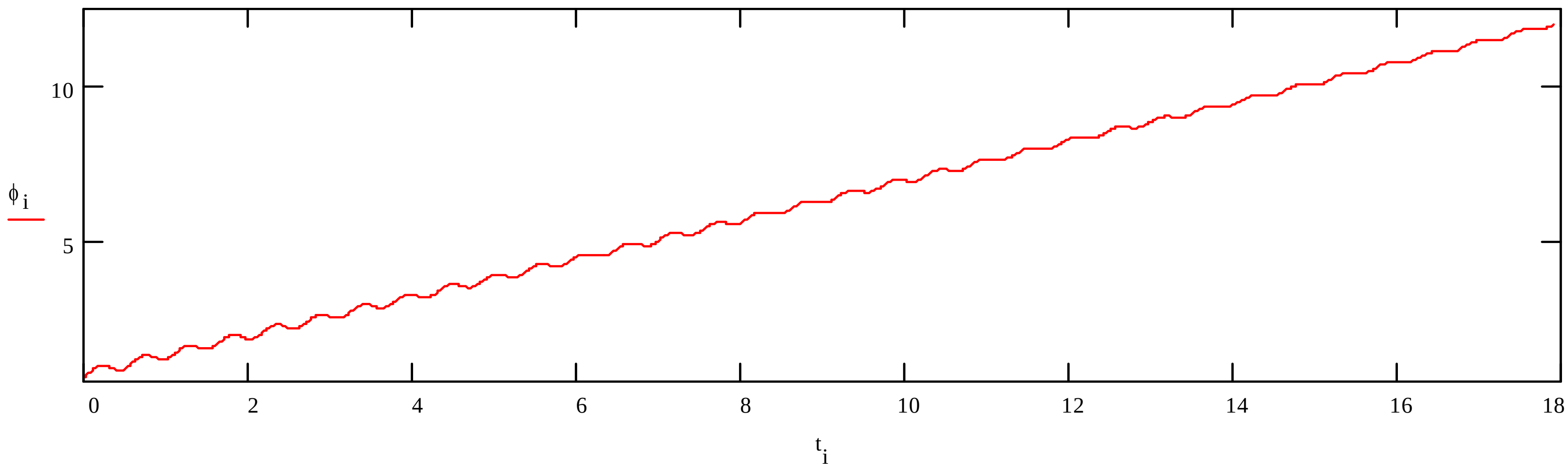
File Edit Window Graph Data Analyze Help
OUT
AUTO
Data Table Window
Row Num
Data Set 1: Data
Ang Torque
1 2.030751 0.028540
2 2.114403 0.028532
3 2.142935 0.028530
4 4.280444 0.056693
5 4.282423 0.056693
6 4.321570 0.056686
7 6.432082 0.084485
8 6.468430 0.084476
9 6.422116 0.084488
10 8.503140 0.111936
11 8.544812 0.111922
12 8.516826 0.111931
13 10.585631 0.139026
14 10.653276 0.138997
15 10.486041 0.139069
16
17
18
19
20
Text Window
Graph Window
Torque vs. Angular Acceleration
Regression:
M = 0.0130356 /, B = 0.0008928
COR = 1.000
Torque
0.15
0.10
0.05
0.00
0
2
4
6
8
10
12
Angular Acceleration
Double click for Graph Options, or click & drag the mouse to select data.

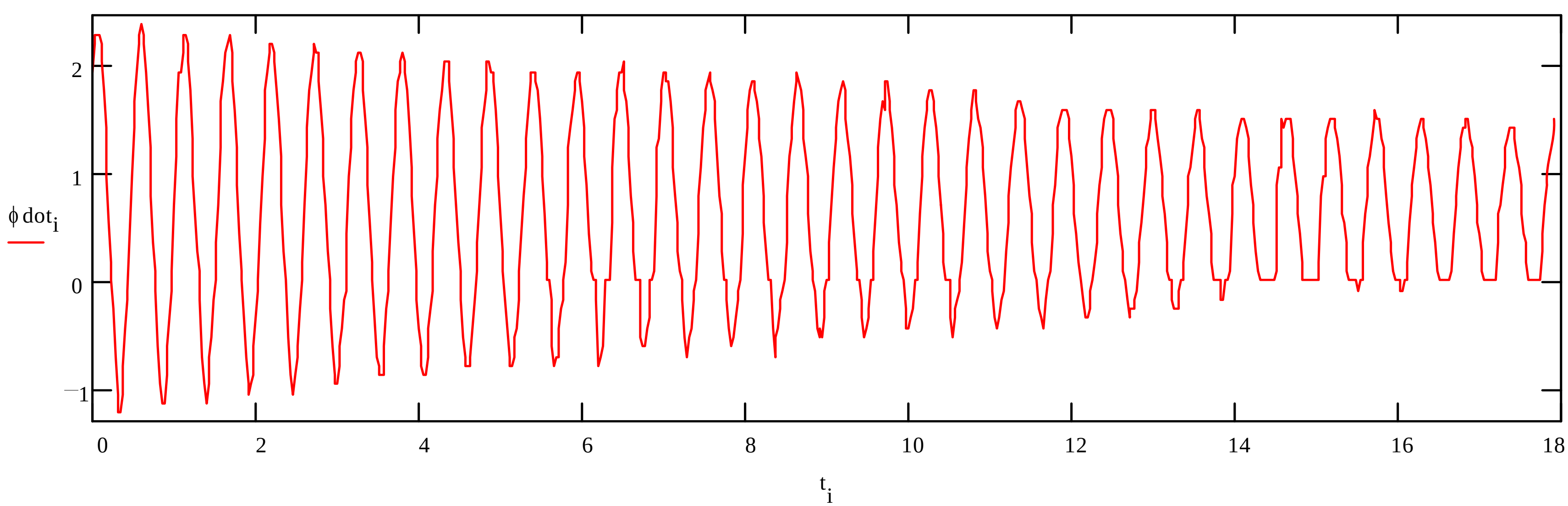
Data for θ and θdot



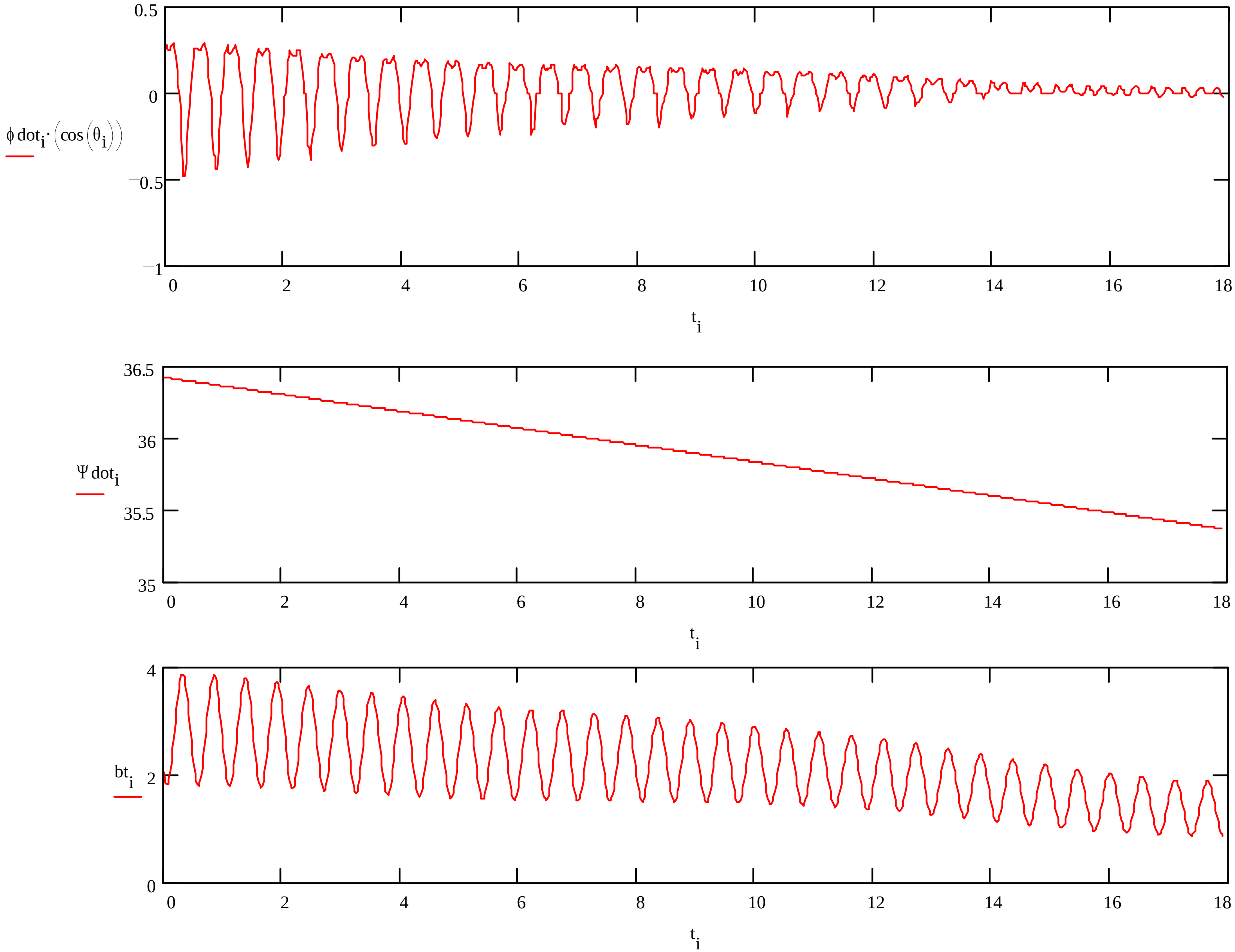


ϕ and ϕdot Data





Secular ψdot Data



ψdot Secular Fit

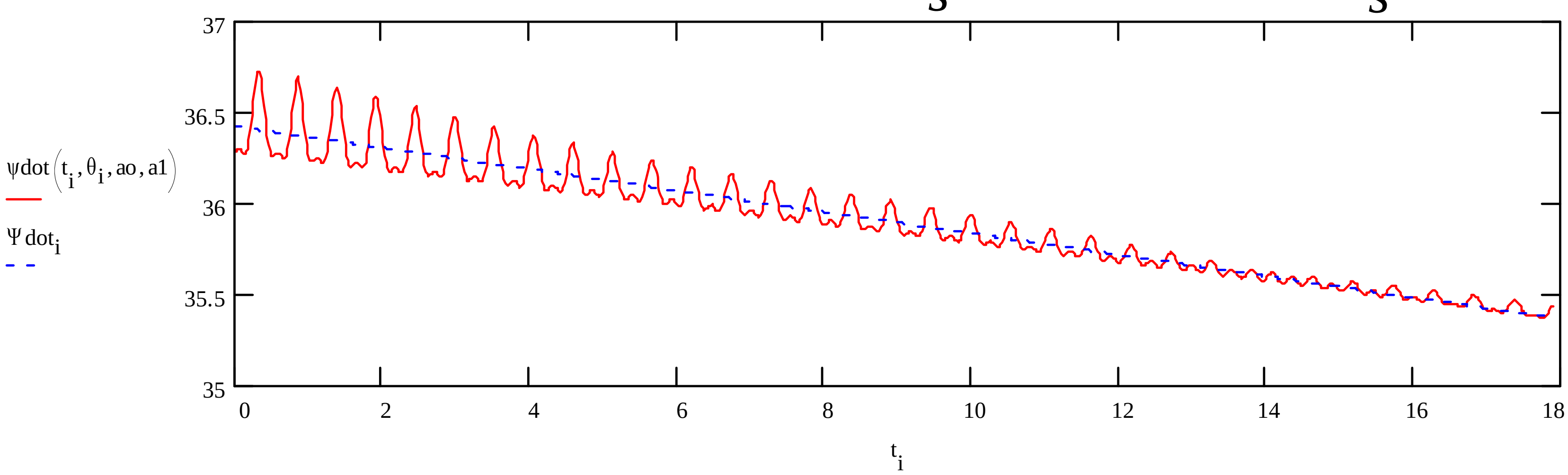
$$\dot{\psi}(0) = 36.4\,\frac{\text{rad}}{\text{s}} \quad \text{and} \quad \alpha = \frac{\Delta\psi}{\Delta t} = -.059\,\frac{\text{rad}}{\text{s}^2}$$

$$I_3(\dot{\psi} + \dot{\phi}\cos\theta) = I_1 a_1\,t + a_o \quad \text{so} \quad a_o \approx \frac{I_3}{I_1}\dot{\psi}(0) = 7.8\,\frac{1}{s} \quad \text{and} \quad a_1 \approx \frac{I_3}{I_1}\alpha = -.012\,\frac{1}{s^2}$$

$$b = b_1\,t + b_o = \left(\sin^2\theta + \frac{I_3}{I_1}\cos^2\theta\right)\dot{\phi} + \frac{I_3}{I_1}\dot{\psi}\cos\theta$$

From the b(t) graph shown earlier $\quad b_o \approx 2.5\,\frac{1}{s} \quad \text{and} \quad b_1 \approx -.07\,\frac{1}{s^2}$

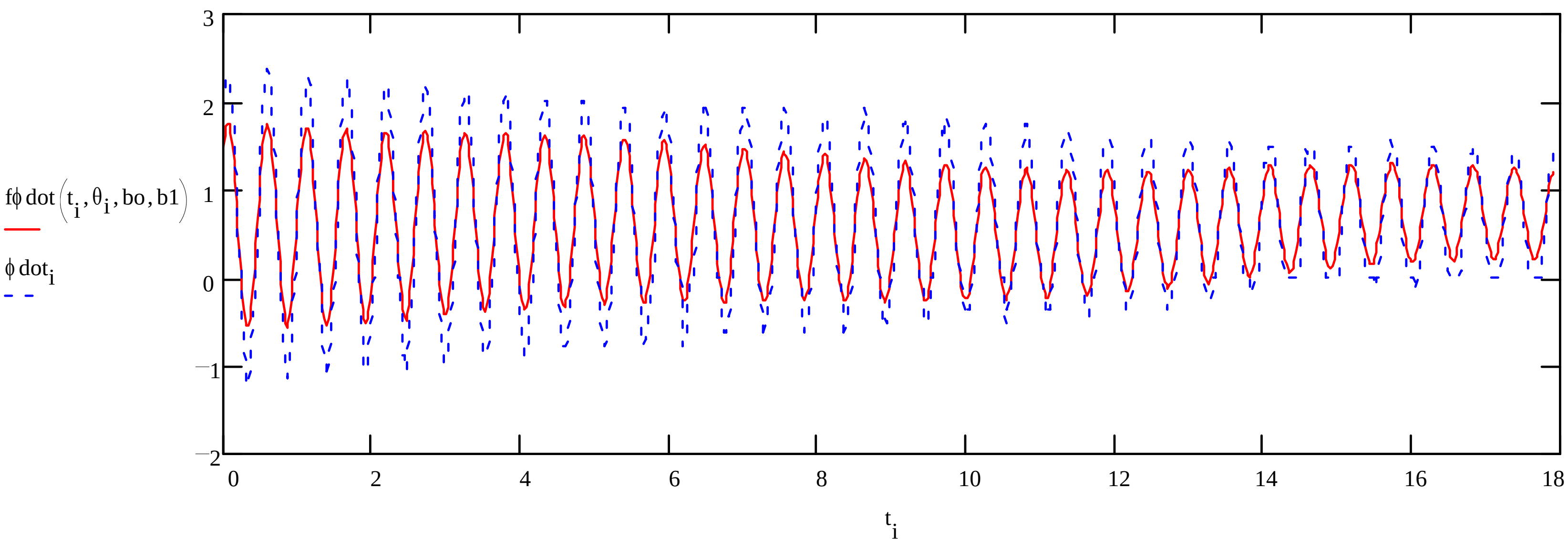
A least squares curve fit yields $\quad a_o = 7.79\,\frac{1}{s} \quad \text{and} \quad a_1 = -.0128\,\frac{1}{s^2}$



ϕdot Data Fit

Least squares fitting ϕdot data to $\dot{\phi} = \dfrac{b_o + b_1\,t - (a_o + a_1\,t)\cos\theta}{\sin^2\theta}$

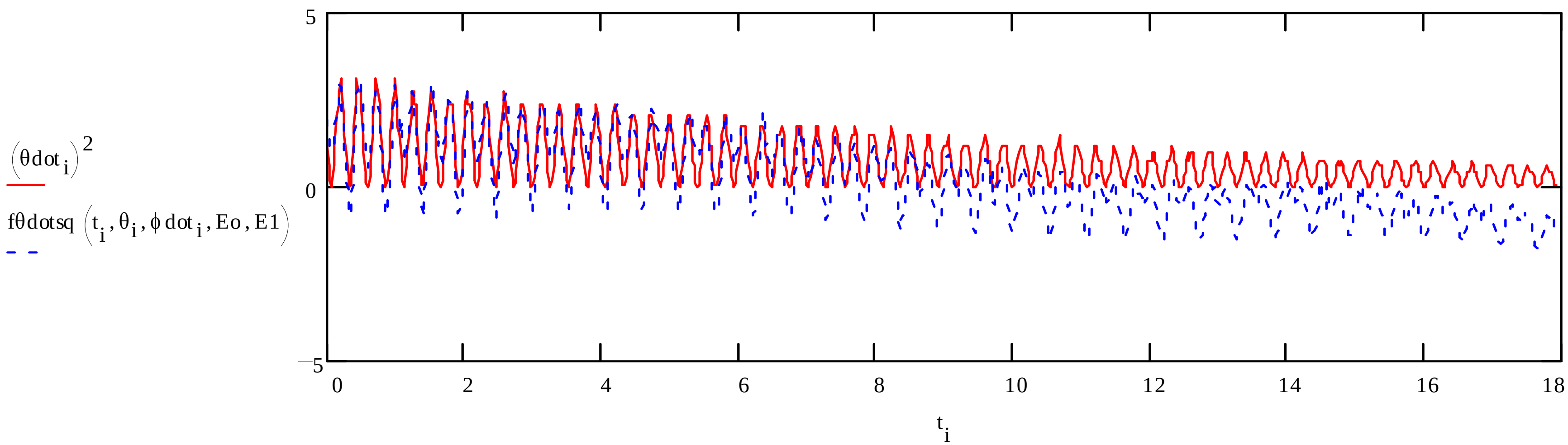
yields $b_o = 2.60\,\dfrac{1}{s}$ and $b_1 = -.085\,\dfrac{1}{s^2}$



Least squares fitting θdot squared data to

$$\dot{\theta}^2 = \frac{2}{I_1}(E_o + E_1 t) - \dot{\phi}^2 \sin^2\theta - \frac{I_1 a^2}{I_3} - \frac{2 M g l \cos\theta}{I_1}$$

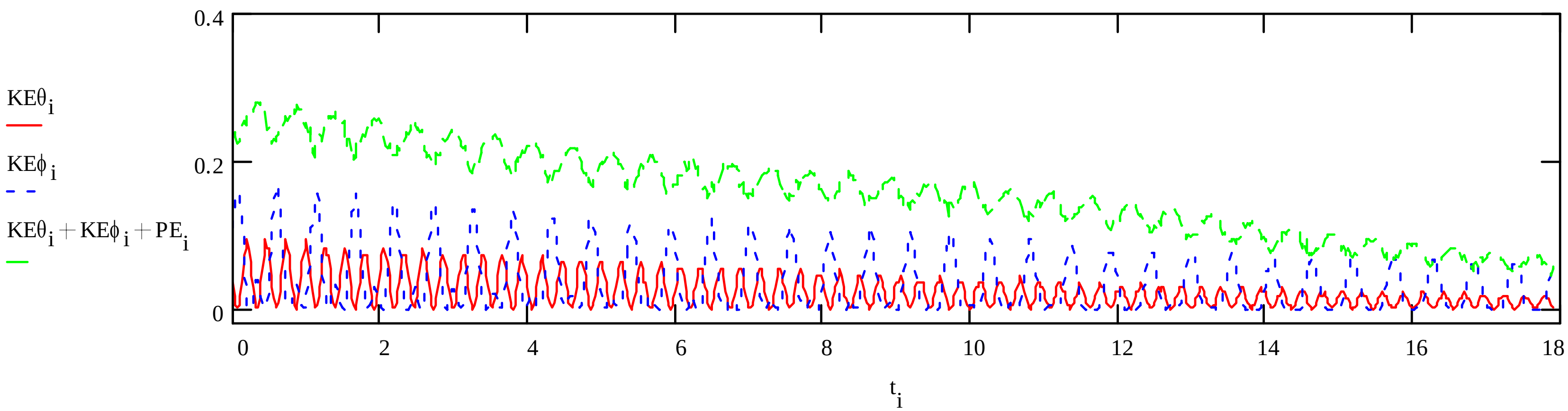
yields $\quad E_o = 8.77\,J \quad$ and $\quad E_1 = -.0356\,\frac{J}{s}$

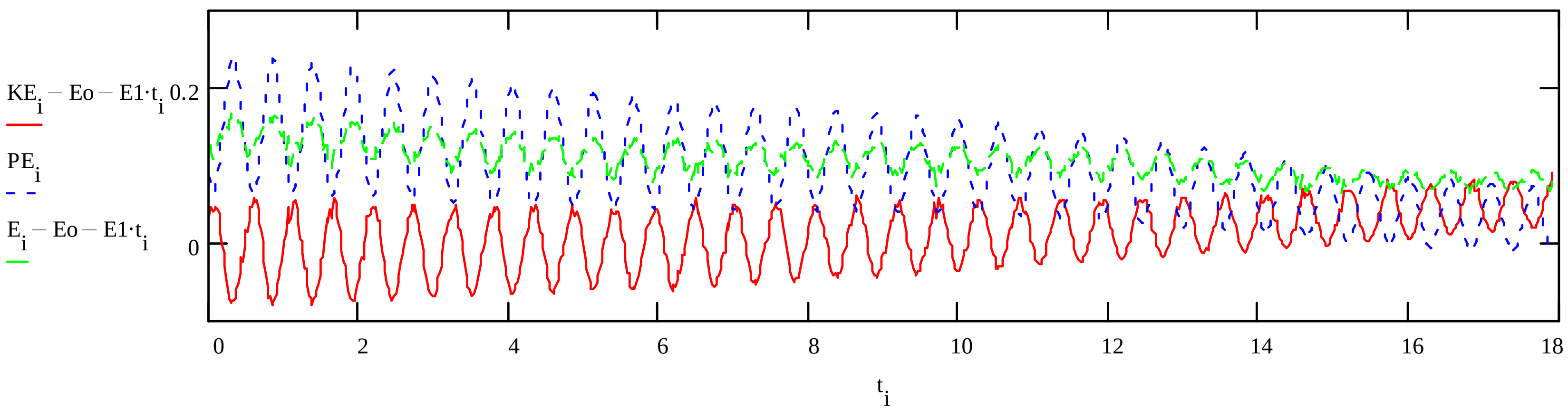


Energies

$$KE_\theta = \frac{I_1}{2}\dot{\theta}^2 \qquad KE_\phi = \frac{I_1}{2}\dot{\phi}^2 \sin^2\theta \qquad KE_\psi = \frac{I_1{}^2}{2\,I_3}(a_o + a_1 t)^2$$

$$PE = Mg\,l\cos\theta$$





Angular Momenta

$$p_\psi = I_3(\dot{\psi} + \dot{\phi}\cos\theta)$$

$$p_\phi = (I_1 \sin^2\theta + I_3\cos^2\theta)\,\dot{\phi} + I_3\,\dot{\psi}\cos\theta$$