Appendix A

Matrices

A.1 Problems

1. $\begin{pmatrix} 1 & -2 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & -3 \end{pmatrix} = \begin{pmatrix} 1(2)-2(1) & 1(-1)-2(-3) \\ -1(2)+0(1) & -1(-1)+0(-3) \end{pmatrix} = \begin{pmatrix} 0 & 5 \\ -2 & 1 \end{pmatrix}$

2. $\begin{pmatrix} 2 & -1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ -1 & 0 \end{pmatrix} =$

3. $\begin{pmatrix} 0 & 2 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix} =$

4. $\begin{pmatrix} 1 & -2 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =$

5. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & -3 \end{pmatrix} =$

6. $\begin{pmatrix} 1 & -2 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1(2)-2(-3) \\ -1(2)+0(-3) \end{pmatrix} = \begin{pmatrix} 8 \\ -2 \end{pmatrix}$

7. $\begin{pmatrix} 1 & -2 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} =$

8. $\begin{pmatrix} 0 & 2t \\ 3t^3 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} =$

9. $\begin{pmatrix} 1 & -2 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 2 & -1 \\ 1 & -3 \end{pmatrix} = \begin{pmatrix} 1+2 & -2+(-1) \\ -1+1 & 0+(-3) \end{pmatrix} = \begin{pmatrix} 3 & -3 \\ 0 & -3 \end{pmatrix}$

10. $\begin{pmatrix} 1 & -2 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 2 & -1 \\ 1 & -3 \end{pmatrix} =$

11. $4\begin{pmatrix} 1 & -2 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 4(1) & 4(-2) \\ 4(-1) & 4(0) \end{pmatrix} = \begin{pmatrix} 4 & -8 \\ -4 & 0 \end{pmatrix}$

12. $3\begin{pmatrix} 1 & -2 \\ -1 & 0 \end{pmatrix} + \alpha \begin{pmatrix} 2 & -1 \\ 1 & -3 \end{pmatrix} =$

13. $\begin{pmatrix} 1 & -2 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2t \\ -t^3 \end{pmatrix} =$

14. $A - \lambda I =$
where $A = \begin{pmatrix} 1 & -2 \\ -1 & 0 \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

15. $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

16. $\begin{vmatrix} 1 & -2 \\ -1 & 0 \end{vmatrix} =$

17. $\begin{vmatrix} 5 & -1 \\ 3 & 6 \end{vmatrix} =$

18. $\begin{vmatrix} (2-\lambda) & 1 \\ -1 & (-1-\lambda) \end{vmatrix} =$

19. Find $\det(A - \lambda I) = |A - \lambda I|$
where $A = \begin{pmatrix} 5 & -1 \\ 3 & 6 \end{pmatrix}$. This is called the *characteristic polynomial* of A.

20. Solve this equation for $\lambda : \det(A - \lambda I) = 0$
where $A = \begin{pmatrix} 5 & -1 \\ 0 & 6 \end{pmatrix}$. These λ-solutions are called *eigenvalues* of the matrix A.

21. Solve this equation for $\lambda : \det(A - \lambda I) = 0$
where $A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$. That is, find the eigenvalues of A.

22. Find the eigenvalues of $A = \begin{pmatrix} -3 & 2 \\ -2 & 2 \end{pmatrix}$.

23. For each of the eigenvalues in problem 22, find a vector $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ (*eigenvector*) such that $Av = \lambda v$; that is, such that v satisfies the equation $(A - \lambda I)v = \mathbf{0}$, where $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the zero vector.

24. Find the eigenvalues and their associated eigenvectors for the matrix $A = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$.

25. Find the eigenvalues and their associated eigenvectors for the matrix $A = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}$.

A.2 Review of linear algebra terms

The λ-polynomial $\det(A - \lambda I)$ is called the *characteristic polynomial* of A.

The λ-equation $\det(A - \lambda I) = 0$ is called the *characteristic equatione.*

The λ-solutions of the characteristic equation $\det(A - \lambda I) = 0$ are called *eigenvalues.*

The vector equation $(A - \lambda I)v = \mathbf{0}$ is called the *eigenvalue problem.*

Vector solutions v of the eigenvalue problem $(A - \lambda I)v = \mathbf{0}$ are called *eigenvectors belonging to* λ.

A.3 Solutions to selected problems

2. $\begin{pmatrix} 2 & -1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 4 & -2 \end{pmatrix}$

7. $\begin{pmatrix} 1 & -2 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - 2y \\ -x \end{pmatrix}$

10. $\begin{pmatrix} 1 & -2 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 2 & -1 \\ 1 & -3 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ -2 & 3 \end{pmatrix}$

12. $3\begin{pmatrix} 1 & -2 \\ -1 & 0 \end{pmatrix} + \alpha \begin{pmatrix} 2 & -1 \\ 1 & -3 \end{pmatrix} = \begin{pmatrix} 3 + 2\alpha & -6 - \alpha \\ -3 + \alpha & -3\alpha \end{pmatrix}$

14. $A - \lambda I = \begin{pmatrix} (1-\lambda) & -2 \\ -1 & -\lambda \end{pmatrix}$

16. $\begin{vmatrix} 1 & -2 \\ -1 & 0 \end{vmatrix} = (1)\,(0) - (-1)(-2) = -2$

18. $\begin{vmatrix} (2-\lambda) & 1 \\ -1 & (-1-\lambda) \end{vmatrix} = -1 - \lambda + \lambda^2$. Note: this polynomial is called the "characteristic polynomial" of A.

19. $\det(A - \lambda I) = \lambda^2 - 11\lambda + 33$ (characteristic polynomial)

20. $\det(A - \lambda I) = 0$ iff $(\lambda - 5)(\lambda - 6) = 0$ iff $\lambda = 5, 6$. Note: The solutions of the equation $\det(A - \lambda I) = 0$ are called the "eigenvalues" of A. The eigenvalues for this matrix A are $\lambda = 5, 6$.

22. $\lambda = 1, -2$ (eigenvalues)

23. $(A - \lambda I)v = \mathbf{0}$ iff $\begin{pmatrix} -3-\lambda & 2 \\ -2 & 2-\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ iff

$$\begin{cases} (-3-\lambda)\,v_1 + 2v_2 = 0 \\ -2v_1 + (2-\lambda)\,v_2 = 0 \end{cases}$$

For $\lambda = 1$ this becomes:

$$\begin{cases} -4v_1 + 2v_2 = 0 \\ -2v_1 + v_2 = 0 \end{cases}$$

Thus, $v_2 = 2v_1$, so choose $v_1 = 1$ and then $v_2 = 2$. For $\lambda = -2$:

$$\begin{cases} -v_1 + 2v_2 = 0 \\ -2v_1 + 4v_2 = 0 \end{cases}$$

Thus, $v_2 = \frac{1}{2}v_1$, so choose $v_1 = 2$ and then $v_2 = 1$. Then an eigenvector belonging to eigenvalue $\lambda = 1$ is $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, while an eigenvector belonging to eigenvalue $\lambda = -2$ is $v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.