

Part II

Discrete Time Models

Chapter 3

Discrete-time Maps

3.1 Time series

Think about almost any area of biology – physiology, genetics, development, ecology, or evolution. Could we study this area without considering time? Certainly not. Look at any biology text or journal. How much could you learn if you ignored all the time-based graphs? Not much. Life happens in time. Biologists, then, concern themselves with many time-based, or *temporal*, aspects of life.

Not surprisingly, mathematical modeling in biology often involves equations that predict temporal processes. A numerical or graphical representation of the state of a system as a function of time is called a *time series*. Both observational data and model predictions can be displayed as time series. For example, Fig. 2.2 depicts two models of humerus growth in glaucous-winged gulls. In each graph, both the observational data (squares and circles) and the model prediction (line) are plotted against time (age in days) and are therefore time series.

Some biological processes, like bone growth, occur continuously in time. Others, like population growth in annual plants, occur more discretely in time. Even though data collected from continuous processes are by necessity taken at discrete time intervals, model predictions based on these data assume temporal continuity and are commonly written in the form of differential equations. By contrast, discrete-time processes are modeled using difference equations, equations that take into account the discontinuous nature of these processes. In this chapter we focus on the modeling of discontinuous biological processes using difference equations.

3.2 Compartmental models

Let x_t be the state of a system—say, the size of a population—at time t . Think of the population as a compartment having inflows and outflows. Inflows are due

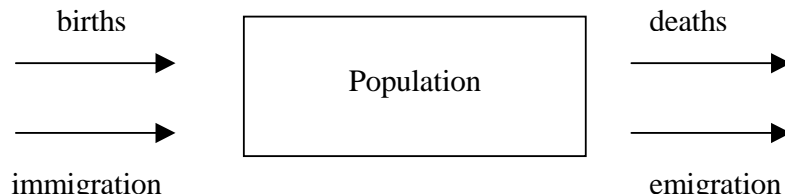


Figure 3.1: A population conceptualized as a compartment with inflows and outflows.

to births, immigration, etc. Outflows are due to deaths, emigration, harvesting, etc. (Fig. 3.1).

In general, *compartment models* have the form

$$\text{net rate of change of quantity in compartment} = \sum \text{inflow rates} - \sum \text{outflow rates.}$$

For continuous time processes, the rate of change is the derivative dx/dt . What is the analogous concept for a discrete-time process? In mathematics, the uppercase Greek letter Δ (delta) usually stands for “change”. We define Δx_{t+1} to be the amount x changes from time t to time $t + 1$. That is,

$$\Delta x_{t+1} = x_{t+1} - x_t.$$

Now, in general, Δx_{t+1} is not a rate of change; it is simply an amount of change. However, in the context of discrete time steps, Δx_{t+1} is the rate of change *per time step*.

Thus, a discrete-time compartmental model has the form

$$\Delta x_{t+1} = \sum (\text{inflow rates at time } t) - \sum (\text{outflow rates at time } t). \quad (3.1)$$

where each flow rate on the right hand side of (3.1) is replaced by a mathematical expression based on the modeling assumptions.

3.3 Linear maps

3.3.1 Malthusian growth

Let x_t be the number of individuals in a population at time t . Assume

- A1) The average *per capita* fecundity (birth rate) is constant and positive.
- A2) Generations are nonoverlapping. All deaths occur at the end of the generational time period.

- A3) The only flows into or out of the population are due to births and deaths.

We need a parameter for the per capita birth rate. Let

$$b = \text{number of births per individual per unit time.}$$

Since the per capita birth rate is b , the total population birth rate will be b times the number of individuals in the population:

$$bx_t = \text{number of births in the population per unit time at time } t.$$

The population death rate is exactly x_t individuals per unit time because all individuals die at the end of each time step. Following equation (3.1), we can write

$$\begin{aligned} \Delta x_{t+1} &= \text{birth rate at time } t - \text{death rate at time } t & (3.2) \\ &= bx_t - x_t. \end{aligned}$$

By definition of Δx_{t+1} , equation (3.2) becomes

$$x_{t+1} - x_t = bx_t - x_t$$

or simply

$$x_{t+1} = bx_t. \quad (3.3)$$

That is, the population size at time $t + 1$ is the per capita birth rate b times the population size at time t .

Suppose the number of individuals at time $t = 0$ is $x_0 = p \geq 0$. The statement $x_0 = p$ is called an *initial value* or *initial condition* for equation (3.3). The population size is therefore modeled by the *initial value problem* corresponding to assumptions A1-A3:

$$\begin{aligned} x_{t+1} &= bx_t \\ x_0 &= p. \end{aligned}$$

The difference equation $x_{t+1} = bx_t$ is called *linear* because the right hand side is a linear function of the state variable x_t .

Example 1 Suppose a petri dish culture initially consists of 10 cells, and that the cells divide into two daughter cells every hour. Then $p = 10$ and $b = 2$, so the culture can be modeled by the initial value problem

$$\begin{aligned} x_{t+1} &= 2x_t \\ x_0 &= 10 \end{aligned}$$

where the time step is one hour. Note that

$$\begin{aligned} x_0 &= 10 = 2^0(10) \\ x_1 &= 2x_0 = 2^1(10) \\ x_2 &= 2x_1 = 2[2(10)] = 2^2(10) \\ x_3 &= 2(2[2(10)]) = 2^3(10) \\ &\vdots \\ x_t &= 2^t(10) \end{aligned}$$

The function $x_t = 10(2^t)$ is called the closed form solution of the initial value problem. “Closed form” means that, given any value of t , you can compute the population size x_t without iterating. Here we see an example of the fact that linear dynamical systems have exponential solutions. Systems that grow exponentially are also said to grow geometrically, or to exhibit Malthusian growth.

In general, the solution of the initial value problem

$$\begin{aligned} x_{t+1} &= bx_t \\ x_0 &= p \end{aligned} \tag{3.4}$$

is

$$x_t = pb^t$$

(exercise 2). Note that the extinction state $x = 0$ is an equilibrium of model (3.4), since it satisfies the equilibrium equation

$$x_e = bx_e.$$

If the initial condition is $x_0 = p = 0$, then the solution is the constant solution $x_t = 0$ for all t . Now suppose $p > 0$. In this case, the behavior of the solution depends on the value of b . If $b > 1$, the population grows exponentially as b^t grows without bound. Thus, the equilibrium state $x_e = 0$ is unstable. If $b = 1$, the population remains at its initial size p for all time. Every solution is an equilibrium solution in this case, and all the equilibria are neutrally stable. If $0 < b < 1$, the population declines exponentially to zero as b^t declines, and so the equilibrium $x_e = 0$ is asymptotically stable.

You can see that the most important expression in model (3.4) is the number b ; the value of b determines the long term fate of the population! The parameter b is called the *intrinsic growth rate*. It is also called the *eigenvalue* of equation (3.4).

Example 2 Let x_t be the number of individuals in a population at time t . Assume

- A1) a constant average per capita birth rate of $b > 0$ offspring per individual per unit time;

- A2) overlapping generations;
- A3) a constant average per capita death rate of d deaths per individual per unit time, with $0 < d < 1$;
- A4) the only flows into or out of the population are due to births and deaths.

Then the population birth rate is bx_t offspring per unit time, and the population death rate is dx_t deaths per unit time. Thus,

$$\begin{aligned}\Delta x_{t+1} &= \text{birth rate at time } t - \text{death rate at time } t \\ &= bx_t - dx_t\end{aligned}$$

or

$$x_{t+1} - x_t = bx_t - dx_t.$$

We can write the model in several equivalent forms to aid in various interpretations. For example, we can write

$$\begin{aligned}x_{t+1} &= x_t + bx_t - dx_t \\ \text{new census} &= \text{last census} + \text{births} - \text{deaths}\end{aligned}$$

We can also write

$$\begin{aligned}x_{t+1} &= bx_t + (1 - d)x_t \\ \text{new census} &= \text{recruits} + \text{survivors}\end{aligned}$$

The term bx_t is called the recruitment term. The number $1 - d$ is the survivorship, the probability that an individual will survive one unit of time. We can also write the model as

$$x_{t+1} = (b + 1 - d)x_t$$

that is,

$$x_{t+1} = rx_t$$

where we define the new parameter r to be $r = b + 1 - d$. Note that $r > 0$ since $b > 0$ and $d < 1$. This example is completed in exercise 3.

3.4 Nonlinear maps

Let x_t be the size of a population at time t . Assume

- A1) The average per capita birth rate would be a constant $b > 0$ offspring per individual per unit time if there were no *crowding effects*.
- A2) Because of crowding effects, the average per capita birth rate is reduced by the factor e^{-cx_t} , where $c > 0$ quantifies the strength of the crowding effect. (Note that $0 < e^{-cx_t} < 1$. The functional form e^{-cx_t} often arises mechanistically from assumptions such as random cannibalistic encounters.)

- A3) Generations are nonoverlapping.

By assumptions A1 and A2, the per capita birth rate in the presence of x_t individuals is be^{-cx_t} ; therefore the population birth rate is $be^{-cx_t}x_t$ offspring per unit time, and the model is

$$\begin{aligned}x_{t+1} &= bx_t e^{-cx_t} \\ x_0 &= p\end{aligned}\tag{3.5}$$

with parameters $b, c > 0$. Model (3.5) is the famous *Ricker model*, historically used in fisheries. It is called *nonlinear* because the right hand side is not a linear function of x_t . This simple deterministic model can have incredibly complex dynamics, as we shall see in the next chapter.

Let's find the equilibrium states of the Ricker model. The equilibrium equation is

$$x_e = bx_e e^{-cx_e}.$$

Note that $x_e = 0$ is a solution. If we divide both sides by x_e , we have the equation $1 = be^{-cx_e}$, which yields a nontrivial equilibrium solution $x_e = \frac{\ln b}{c}$.

It is often useful to graph the equilibria as a function of one of the parameters. In this case we graph the two equilibria as a function of b (Fig. 3.2). Note that the nontrivial equilibrium $x_e = \frac{\ln b}{c}$ is positive if and only if $b > 1$. The value of $b = 1$ at which the two equilibrium branches cross is called a *bifurcation point*. The graph of x_e vs b (Fig. 3.2) is called a *bifurcation diagram*.

Note: A model such as $x_{t+1} = bx_t e^{-cx_t}$ is really a collection of infinitely many models, one for each specific pair of values of b and c . When looking at a bifurcation diagram such as Fig. 3.2, you must remember that any particular system has a fixed value of b , for example, $b = 1.5$. Its equilibria are given by the values of the equilibrium branches directly above that specific value of b . Another way to say it is that any particular system “lives on a vertical line” in Fig. 3.2.

What are the stabilities of these two equilibria? Do their stabilities depend on the value of b ? Clearly we need to find a way to quantify and study stability in nonlinear equations. To this end we now turn to the important subject of *linearization*.

3.5 Linearization

3.5.1 Linearization of functions

Suppose $f(x)$ is a function whose graph passes through the point $(a, f(a))$. Close to the point $(a, f(a))$, we can approximate the graph of f with a straight line that is tangent to the graph of f at the point $(a, f(a))$. What is the equation of this line? Recall that the slope m of a line is the rise over the run. Given a nearby point (x, y) on the line (Fig. 3.3), the rise from $(a, f(a))$ to (x, y) is $y - f(a)$, while the run is $x - a$. The slope is therefore

$$m = \frac{y - f(a)}{x - a},$$

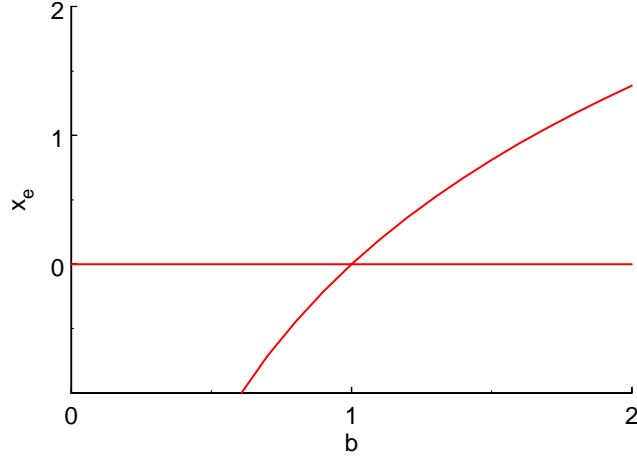


Figure 3.2: Bifurcation diagram for the Ricker map shows the equilibria x_e as a function of the parameter b . The two equilibrium branches are $x_e = 0$ and $x_e = (\ln b)/c$.

and so

$$y - f(a) = m(x - a).$$

From Calculus, we know that the slope m of the line tangent to f at the point $(a, f(a))$ is the derivative of f evaluated at $x = a$:

$$m = \frac{df}{dx}(a).$$

Thus, the equation of the tangent line is

$$y - f(a) = \left[\frac{df}{dx}(a) \right] (x - a),$$

or

$$y = f(a) + (x - a) \frac{df}{dx}(a). \quad (3.6)$$

Equation (3.6) is called the *linearization of f at $x = a$* .

Note: You always have to linearize about a point. It doesn't make sense to ask what the linearization of a function is without reference to a specific point $x = a$. Furthermore, the linearization of a function is always a linear equation of the form $y = mx + b$, where m and b are constants.

Because the linearization is a good approximation to the function f near the point $x = a$, we have

$$f(x) \approx f(a) + (x - a) \frac{df}{dx}(a) \text{ for } x \approx a.$$

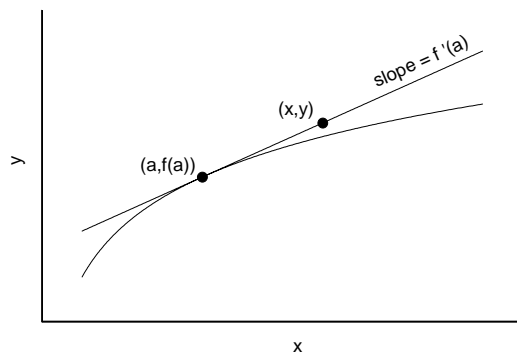


Figure 3.3: A function can be approximated by a tangent line

Linearization is one of the most important tools in applied mathematics, and is extremely important in modeling. In fact, linearization provides the key to studying stability, as we shall soon see.

Example 3 Find the linearization of the curve $f(x) = x^3 - 4x^2 + 7x - 1$ at $x = 2$. *Solution:* In this problem we have $a = 2$. The derivative of f is

$$\frac{df}{dx} = 3x^2 - 8x + 7.$$

Thus,

$$f(a) = f(2) = 2^3 - 4(2)^2 + 7(2) - 1 = 5$$

and

$$\begin{aligned} \frac{df}{dx}(a) &= \frac{df}{dx}(2) \\ &= 3(2)^2 - 8(2) + 7 \\ &= 3. \end{aligned}$$

Hence the linearization is

$$\begin{aligned} y &= f(2) + (x - 2) \frac{df}{dx}(2) \\ &= 5 + (x - 2)(3) \\ &= 3x - 1, \end{aligned}$$

and therefore we have the approximation

$$f(x) \approx 3x - 1 \text{ for } x \approx 2.$$

3.5.2 Linearization of discrete-time maps

Consider the nonlinear discrete dynamical system

$$x_{t+1} = f(x_t) \tag{3.7}$$

with equilibrium x_e . When we say equation (3.7) is *nonlinear*, we mean that f is a nonlinear function of x_t . An example of such an equation would be

$$x_{t+1} = ax_t^2.$$

What we would like to do is replace $f(x_t)$ on the right hand side of (3.7) with a simpler linear function that approximates it for values of $x_t \approx x_e$.

In the previous section we learned that

$$f(x) \approx f(a) + \left[\frac{df}{dx}(a) \right] (x - a) \text{ for } x \approx a.$$

In the current context, x is x_t , and a is x_e . Making those substitutions, we have

$$f(x_t) \approx f(x_e) + \left[\frac{df}{dx}(x_e) \right] (x_t - x_e) \text{ for } x_t \approx x_e.$$

Also, since x_e is an equilibrium, we know that $x_e = f(x_e)$.

Note: The constant x_e is an equilibrium of the difference equation $x_{t+1} = f(x_t)$ if and only if it satisfies the equilibrium equation $x_e = f(x_e)$.

Thus, we can write

$$f(x_t) \approx x_e + \left[\frac{df}{dx}(x_e) \right] (x_t - x_e) \text{ for } x_t \approx x_e.$$

It is traditional to define the Greek letter λ (lambda) to be the derivative df/dx evaluated at x_e :

$$\lambda = \left[\frac{df}{dx}(x_e) \right].$$

Note: $\lambda = \left[\frac{df}{dx}(x_e) \right]$ is simply a constant, a number that you can compute.

Then

$$\begin{aligned} x_{t+1} &= f(x_t) \\ &\approx x_e + \lambda(x_t - x_e) \text{ for } x_t \approx x_e, \end{aligned}$$

and hence

$$x_{t+1} - x_e \approx \lambda(x_t - x_e) \text{ for } x_t \approx x_e. \tag{3.8}$$

We can restate equation (3.8) in words: As long as the system is in the neighborhood of the equilibrium, the displacement of the system state from equilibrium at time $t + 1$ is approximately the number λ times the displacement of the system from equilibrium at time t . That is, x_{t+1} is about λ times as far from the equilibrium as x_t was. You can see that if $0 < \lambda < 1$, this is good news for the stability of x_e , because the distance between the system state and the equilibrium is shrinking as time goes on. In this case, it appears that the equilibrium x_e would be asymptotically stable. If $\lambda > 1$ or $\lambda < -1$, however, the distance between the system state and the equilibrium grows whenever the system is near the equilibrium, and so it appears that x_e would be unstable.

We now clean up the notation a bit and state these observations in a theorem. Define

$$\begin{aligned} z_t &= x_t - x_e & (3.9) \\ &= \text{displacement, or variation, of the system state from equilibrium.} \end{aligned}$$

Using the change of variables (3.9) we can rewrite equation (3.8) as

$$z_{t+1} \approx \lambda z_t \text{ for } z_t \approx 0. \quad (3.10)$$

When the \approx is replaced by $=$, equation (3.10) is called the *variation equation*. The change of variables (3.9) simply shifts the equilibrium to zero. Note that the variation equation is linear, and its solution is

$$z_t = z_0 \lambda^t.$$

Thus, the displacement of the population from equilibrium grows or decays exponentially *when the population size is near its equilibrium value*. Whether the displacement grows or decays depends on whether the eigenvalue λ is greater than or less than 1 in absolute value.

Definition 4 The *linearization of $x_{t+1} = f(x_t)$ at the equilibrium x_e* is the linear map

$$z_{t+1} = \lambda z_t$$

where the *eigenvalue* λ is given by

$$\lambda = \frac{df}{dx}(x_e).$$

Note: You must always linearize a difference equation about a specific equilibrium. It does not make sense to “find the linearization” without reference to a particular equilibrium. Furthermore, the linearization of a nonlinear difference equation is always a linear difference equation of the form $z_{t+1} = \lambda z_t$, where λ is a number.

Definition 5 An equilibrium x_e of $x_{t+1} = f(x_t)$ is *hyperbolic* if and only if $|\lambda| \neq 1$.

Theorem 6 (Linearization Theorem) Let $x_{t+1} = f(x_t)$ be a nonlinear map with hyperbolic equilibrium x_e . Suppose also that the function f is continuously differentiable in x (that is, df/dx exists and is continuous). Then

$$\begin{aligned} |\lambda| < 1 &\implies x_e \text{ is asymptotically stable} \\ |\lambda| > 1 &\implies x_e \text{ is unstable} \end{aligned}$$

where

$$\lambda = \frac{df}{dx}(x_e).$$

Note: The nonhyperbolic case $|\lambda| = 1$ is not covered by the Linearization Theorem. If $|\lambda| = 1$, the theorem simply does not apply, and no conclusion can be drawn from the theorem.

3.5.3 Linearizing the Ricker map

Consider again the Ricker map

$$x_{t+1} = bx_t e^{-cx_t},$$

where $b, c > 0$. Here $f(x) = bx e^{-cx}$. The equilibria are

$$x_e = 0 \text{ and } x_e = \frac{\ln b}{c}.$$

Let's linearize the Ricker map about each of its equilibria. In both cases we will need the derivative of f with respect to x :

$$\begin{aligned} \frac{df}{dx} &= be^{-cx} - cbx e^{-cx} \\ &= be^{-cx}(1 - cx). \end{aligned}$$

We first linearize about $x_e = 0$. Now,

$$\lambda = \frac{df}{dx}(0) = be^0(1 - 0) = b,$$

and so the linearization at $x_e = 0$ is

$$z_{t+1} = bz_t.$$

Thus,

$$\begin{aligned} 0 < b < 1 &\implies x_e = 0 \text{ is asymptotically stable} \\ b > 1 &\implies x_e = 0 \text{ is unstable.} \end{aligned}$$

Now we linearize about the nontrivial equilibrium $x_e = \frac{\ln b}{c}$:

$$\begin{aligned}\lambda &= \frac{df}{dx} \left(\frac{\ln b}{c} \right) \\ &= be^{-c\left(\frac{\ln b}{c}\right)} \left(1 - c \left(\frac{\ln b}{c} \right) \right) \\ &= be^{\ln b^{-1}} (1 - \ln b) \\ &= bb^{-1} (1 - \ln b) \\ &= 1 - \ln b.\end{aligned}$$

In this case the linearization at $x_e = \frac{\ln b}{c}$ is

$$z_{t+1} = (1 - \ln b) z_t,$$

and so

$$\begin{aligned}|1 - \ln b| < 1 &\implies x_e = \frac{\ln b}{c} \text{ is asymptotically stable} \\ |1 - \ln b| > 1 &\implies x_e = \frac{\ln b}{c} \text{ is unstable.}\end{aligned}$$

If we do some algebra, we see that

$$\begin{aligned}1 < b < e^2 &\implies x_e = \frac{\ln b}{c} \text{ is stable} \\ 0 < b < 1 &\implies x_e = \frac{\ln b}{c} \text{ is unstable} \\ b > e^2 &\implies x_e = \frac{\ln b}{c} \text{ is unstable.}\end{aligned}$$

We can now indicate stability on our bifurcation diagram (Fig. 3.4):

Note that if $b > e^2$, both equilibria are unstable! How do the solutions of the Ricker map behave for these values of b ? Toward what value do they tend, if any? What is the long term fate of the system? We will investigate this fascinating question in Chapter @.

3.6 Exercises

1. A population of annual plants has an average per capita fecundity of 120.3 seeds per plant per year. Approximately 12% of these seeds germinate. About 10% of the resulting seedlings survive until maturity. The initial population census (year $t = 0$) is 52 plants.
 - (a) How can the fecundity be 120.3? What does 0.3 seeds mean, anyway?
 - (b) Write down the population model (the initial value problem).

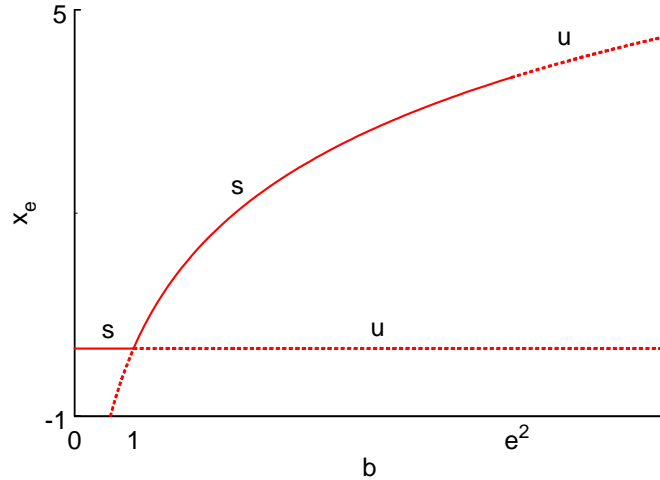


Figure 3.4: Bifurcation diagram of the Ricker map showing stability of equilibria. Dotted equilibria are unstable; solid are stable.

- (c) Find the closed-form solution of the initial value problem.
 - (d) Will the population grow or decline? Explain.
 - (e) Is the extinction state stable or unstable?
 - (f) How many plants will there be at year $t = 5$, according to the model?
 - (g) How many years will it take for the population to grow to 75 plants, according to the model?
 - (h) How many years will it take for the population to double its initial size, according to the model? This is called the *doubling time*.
 - (i) Prove that the doubling time in this model does not depend on the initial population size.
2. (a) Show that the solution of model (3.4) is $x_t = pb^t$.
 - (b) If $p > 0$ and $0 < b < 1$, what is $\lim_{t \rightarrow \infty} x_t$? What does this mean in biological terms?
 - (c) If $p > 0$ and $b > 1$, what is $\lim_{t \rightarrow \infty} x_t$? What does this mean in biological terms?
 - (d) If $p > 0$ and $b = 1$, what is $\lim_{t \rightarrow \infty} x_t$? What does this mean in biological terms?
 - (e) Write a program that *iterates* the recursion formula (3.4) to produce a time series of length n . In your program, set $p = 50$ and $n = 5$. Graph the time series for each of $b = 0.5, 0.9, 1.0, 1.1,$ and 1.5 .

Display all 5 time series on the same graph. Turn in the program and the graph.

3. Consider the linear population model

$$\begin{aligned}x_{t+1} &= (b + 1 - d)x_t \\ x_0 &= p > 0\end{aligned}$$

where $b > 0$ and $0 < d < 1$.

- (a) Derive the closed form solution.
(b) Use the closed form solution to prove that

$$\begin{aligned}b < d &\implies \lim_{t \rightarrow \infty} x_t = 0 \\ b > d &\implies \lim_{t \rightarrow \infty} x_t = \infty \\ b = d &\implies x_t = p \text{ for all } t\end{aligned}$$

- (c) Give a biological interpretation for the results in (b). Does this make sense biologically?
4. Find the linearization of the function $f(x) = x^4 - x + 15$ at $x = 1$. Graph the function and its linearization (on the same graph) near $x = 1$.
5. Find the linearization of the function $f(x) = \sin x$ at $x = 0$. Graph the function and its linearization (on the same graph) near $x = 0$.
6. In this problem you will carry out a complete equilibrium stability analysis for the Beverton-Holt population model

$$\begin{aligned}x_{t+1} &= \frac{bx_t}{1 + cx_t} \\ x_0 &= p \geq 0 \\ b, c &> 0\end{aligned}$$

- (a) Is this model linear or nonlinear? Why?
(b) Find all the equilibria.
(c) Graph the equilibria x_e as functions of the parameter b .
(d) Under what conditions is the nontrivial equilibrium positive?
(e) Find the linearization of the model at each equilibrium.
(f) For what values of b is each equilibrium stable/unstable?
(g) Indicate the stabilities on your bifurcation diagram in (c).
(h) What are the bifurcation points on your diagram in (c)?

- (i) Write a program that iterates the Beverton-Holt map to produce a time series of length n . In your program, set $p = 15$, $c = 0.002$, and $n = 100$. Graph the time series for each of $b = 0.5, 0.9, 1.0, 1.1$, and 1.5 . Display all 5 time series on the same graph. Turn in the program and the graph.
7. In this problem you will carry out a complete equilibrium stability analysis for the so-called *discrete logistic map*

$$\begin{aligned}x_{t+1} &= rx_t(1 - x_t) \\ 0 &< x_0 < 1 \\ r &> 0\end{aligned}$$

population model. Here x_t is the density of organisms at time t .

- Is this model linear or nonlinear? Why?
- Find all the equilibria.
- Graph the equilibria x_e as functions of the parameter r .
- Under what conditions are the equilibria positive?
- Find the linearization of the model at each equilibrium.
- For what values of r is each equilibrium stable/unstable?
- Indicate the stabilities on your bifurcation diagram in (c).
- What are the bifurcation points on your diagram in (c)?
- Write a program that iterates the discrete logistic map to produce a time series of length n . In your program, set $x_0 = 0.5$ and $n = 30$. Graph the time series for each of $r = 0.5, 1.5, 2.5, 3.2$, and 3.5 . Display all 5 time series on the same graph. Turn in the program and the graph. Also turn in the 5 numerical time series as lists of numbers.
- Explore the behavior of the discrete logistic map for $3.5 < r < 4$. You may have to increase b by small increments to see how the time series change.
- Mathematically, what happens to solutions when $r > 4$? (Hint: By hand, iterate $x_{t+1} = 5x_t(1 - x_t)$ starting with $x_0 = 0.5$. What happens?)

