Math 215 Review 3

This exam covers from Sec 4.4 thru 4.6 and Sec 5.1 thru 5.3 you should review all the related materials for the exam. You are not allowed to use a graphic calculator for the exam.

1. A is a 3 by 3 matrix with eigenvectors $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ corresponding to eigenvalues $\lambda_1 = -\frac{1}{3}$, $\lambda_2 = \frac{1}{3}$ and $\lambda_3 = 1$, respectively and $\vec{x} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$. Find $A^{20}\vec{x}$ and $A^k\vec{x}$ if k becomes large?

2. (a) Is A in problem 1 diagonalizable? Why or why not?

(b) Find $A^{20}$.

3. A market research team is conducting a survey to determine people's preferences in toothpaste. The sample consists of 200 people, each of whom is asked to try both brands of toothpaste over a period of several months. Of those using brand A in any month, 70% continue to use it the following month, while 30% switch to brand B; of those using brand B in any month, 80% continue to use it the following month, while 20% switch to brand A. Suppose that when the survey begins, 120 people are using brand A and 80 people are using brand B.

(a) How many people will be suing each brand 1 month later? 2 months later?

(b) In the long run, what can you say about this survey?

4. Solve the system: $\begin{pmatrix} x' = 3x + 4y \\ y' = 3x + 2y \end{pmatrix}$.

5. (True or false) If true, prove it. Otherwise, give a counterexample. Don't simply say by theorem.

(a) If $\{v_1, v_2, ..., v_n\}$ is linearly independent set of vectors, then it is orthogonal sets.

(b) Every diagonalizable matrix is invertible.

(c) If Q is an orthogonal matrix, then $\det Q = \pm 1$.

(d) If $Q_1$ and $Q_2$ are orthogonal matrices, then $Q_1Q_2$ is an orthogonal matrix as well.

6. Write the vector $x = \begin{pmatrix} 7 \\ -3 \\ 2 \end{pmatrix}$ as a linear combination of the orthogonal basis $\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}\}$.

7. Find a basis for $W^{\perp}$ if $W = \text{span}\{\begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}\}$.

8. (a) Apply the Gram-Schmidt process to $\{x_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, x_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, x_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}\}$ to find an orthogonal basis for W.

(b) Use (a) to find a QR factorization of A= $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$.

(c) Use a QR factorization in (b) to solve the system $A\vec{x} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$.

Answer key to review 3 Math 215

1. $\begin{pmatrix} 2 \\ 2-(1/3)^{20} \\ 2 \end{pmatrix}$, $A^k\vec{x} = \begin{bmatrix} 2-((-1)^k-1)/3^k \\ 2-1/3^k \\ 2 \end{bmatrix}$ so that $A^k x \to \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$ as k $\to \infty$ .

2. (a) Yes because it has three distinct eigenvalues which means there are three linearly independent eigenvectors.

(b) $P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, $D = \begin{pmatrix} -1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $P^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$. Then

$$A^{20} = PD^{20}P^{-1} = \begin{pmatrix} 1/3^{20} & 0 & -1/3^{20}+1 \\ 0 & 1/3^{20} & -1/3^{20}+1 \\ 0 & 0 & 1 \end{pmatrix}$$

3. (a) $\begin{pmatrix} 100 \\ 100 \end{pmatrix}$, $\begin{pmatrix} 90 \\ 110 \end{pmatrix}$ (b) $\begin{pmatrix} 80 \\ 120 \end{pmatrix}$

4. $\begin{pmatrix} 4c_1e^{6t}+c_2e^{-t} \\ 3c_1e^{6t}-c_2e^{-t} \end{pmatrix}$

5. (a) F (b) F (c) T (d) T

6. $(9/2, 2/3, -11/6)$

7. $\{\begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}\}$

8. (a) $\frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\frac{1}{\sqrt{12}}\begin{pmatrix} 1 \\ 1 \\ 1 \\ -3 \end{pmatrix}$, $\frac{1}{\sqrt{6}}\begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

(b) $Q = \begin{pmatrix} 1/2 & 1/\sqrt{12} & -2/\sqrt{6} \\ 1/2 & 1/\sqrt{12} & 1/\sqrt{6} \\ 1/2 & 1/\sqrt{12} & 1/\sqrt{6} \\ 1/2 & -3/\sqrt{12} & 0 \end{pmatrix}$ and $R = \begin{pmatrix} 2 & 3/2 & 3/2 \\ 0 & \sqrt{3}/2 & -\sqrt{3}/6 \\ 0 & 0 & \sqrt{6}/3 \end{pmatrix}$.

(c) (1,-1,1)

This is a brief outline of the main topics we had in class.

Sec 4.4 Similarity and Diagonalization
Similar matrix, Several properties for two similar matrices (Theorem 4.22), Diagonalizable, Theorem 4.23, and 4.25, 4.26, 4.27, Convert the matrix $A$ to $PDP^{-1}$, find $A^n$ using the diagonalization

Sec 4.5 & 4.6 Applications
Markov Chains, Applications problem requiring a higher power of matrix A, System of first order differential equations

Sec 5.1 Orthogonality in $\mathbb{R}^n$
Orthogonal set, Theorem 5.1, Orthogonal (Orthonormal) basis, Theorem 5.2, Orthogonal matrix, Theorem 5.4 through 5.8

Sec 5.2 Orthogonal Complement and Orthogonal projections
Orthogonal Complements, Theorem 5.9, 5.10, Fundamental subspaces of the matrix A, Orthogonal projection of v onto a subspace W, Orthogonal Decomposition theorem(Theorem 5.11), Corr 5.12 and 5.13 and 5.14

Sec 5.3 The Gram-Schmidt Process and the QR factorization
The Gram-Schmidt process, The QR factorization, solve the system using QR factorization