Math 215, Introduction to Linear Algebra
Final Exam Study Guide

You should FIRST study all exams and quizzes we have done this semester. Then go over worksheets, homework problems and examples done in class.
The final is comprehensive and it is worth 20% of overall grade so that it will change your grade significantly. To maintain or improve your grade, it is crucially important to do well on the final.

The general topics are listed below.

Gauss-Jordan Elimination
Span and linear independency
Matrix operations, inverses, elementary matrices
The fundamental theorem of inverse matrices
LU factorization
Subspaces, basis, rank, dimension
Linear transformations, matrix representation, homogeneous coordinates
Determinants, Cramer's rule
Eigenvalues and eigenvectors
Similarity and diagonalization
Orthogonalization and Gram-Schmidt, QR factorization
Least Squares

Applications: Network analysis, Iterative methods, Markov chains, graphs and digraphs, System of linear differential equations

1. (a) Let $A=\begin{pmatrix} 2 & -8 \\ 1 & -4 \end{pmatrix}$. Find an invertible matrix P and a diagonal matrix D with $A=PDP^{-1}$. Verify your answer.
(b) **<u>Use (a)</u>** to find A^{25}.
2. Solve the initial value problem $Y'=AY, Y(0)=Y_0$ where $A=\begin{pmatrix} 3 & 4 \\ 3 & 2 \end{pmatrix}, Y_0=\begin{pmatrix} 6 \\ 1 \end{pmatrix}$.
3. Prove the following.
(a) Let A be a diagonalizable matrix whose eigenvalues are all either 1 or -1. Show that $A^{-1}=A$.
(b) An n by n matrix A is invertible iff det(A)$\neq$ 0.
(c) Let L be a linear operator on $\mathbb{R}^n$. Suppose that $L(x)=0$ for some nonzero x. Let A be the matrix representing L with respect to the standard basis $\{e_1, e_2, ..., e_n\}$. Show that A is not invertible.
(d) Let $x_1, ..., x_k$ be linearly independent vectors in $\mathbb{R}^n$, and let A be an invertible n by n matrix. Define $y_i = Ax_i$, for $i=1,...,k$. Show that $y_1,..., y_k$ are linearly independent.
(e) Let A be a square matrix with eigenvalue λ and corresponding eigenvector x. For any positive integer n, λ^n is an eigenvalue of A^n with corresponding eigenvector x.
(f) An n by n matrix A is invertible iff nullity(A) = 0.

4. Answer the following question. Explain.
(a) Let $A=[a_1 \; a_2 \; a_3]$ be a 3 by 3 matrix and let $\mathrm{b}=3\mathrm{a}_1-2\mathrm{a}_2+\mathrm{a}_3$ for column vectors a_i's. Will the system $Ax=\mathrm{b}$ be consistent?

(b) If $|A| = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = 4$, find the determinant of the matrix $\begin{pmatrix} 2a_3 & a_2 & a_1 \\ 2b_3 & b_2 & b_1 \\ 2c_3 & c_2 & c_1 \end{pmatrix}$

(c) Let A be an n by n orthogonal matrix. Find the possible value for its determinant. Explain.

(d) The mapping L from $\mathbb{R}^2$ to $\mathbb{R}^3$ defined by $L(\mathrm{x}) = (x_2, x_1, x_1 + x_2)^T$ is linear.
Find the matrix A representing L with respect to the standard basis.

5. Express the matrix A given to the right as a product of elementary matrices.

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 3 & 8 & 7 \end{pmatrix}$$

6. Compute the Gram-Schmidt QR factorization of the matrix B and use it to find the least squares solution to $\begin{pmatrix} 1 & -2 & -1 \\ 2 & 0 & 1 \\ 2 & -4 & 2 \\ 4 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \\ -2 \end{pmatrix}$.

$$\boxed{B = \begin{pmatrix} 1 & -2 & -1 \\ 2 & 0 & 1 \\ 2 & -4 & 2 \\ 4 & 0 & 0 \end{pmatrix}}$$

7. In an experiment designed to determine the extent of a person's natural orientation, a subject is put in a special room and kept there for a certain length of time. He is then asked to find a way out of a maze and a record is made of the time it takes the subject to accomplish this task. The following data are obtained.

Time in room(hour)	1	2	3	4	5	6
Time to find way(min)	0.8	2.1	2.6	2.0	3.1	3.3

Let x denote the number of hours in the room and let y denote the number of minutes that it takes the subject to find his way out. (a) Find the least squares line relating x and y. (b) Use the equation in (a) to estimate the time it will take the subject to find his way out of the maze after 12 hours in the room.

8. Find the eigenvalues and the corresponding eigenvectors of the matrix $A = \begin{pmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{pmatrix}$.

9. Suppose A is a 2 by 2 matrix with eigenvectors $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ corresponding to eigenvalues $\lambda_1 = \dfrac{1}{2}$ and $\lambda_2 = 2$, respectively and $x = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$. Find $A^{10}x$.

10. Answer the following question.

(a) Let T: $\mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - 2y \\ x + y - 3z \end{pmatrix}$. Find the matrix representation with respect to the standard basis.

(b) Find the inverse of $A=\begin{pmatrix}1 & 2 & -1\\ 2 & 2 & 4\\ 1 & 3 & -3\end{pmatrix}$ if it exists.

(c) Find a basis for the four fundamental subspace of the matrix $\begin{pmatrix}1 & 1 & 3 & 1\\ 2 & -1 & 0 & 1\\ -3 & 2 & 1 & -2\\ 4 & 1 & 6 & 1\end{pmatrix}$.

(d) Do $\begin{pmatrix}1\\ -1\\ 3\end{pmatrix}, \begin{pmatrix}-1\\ 5\\ 1\end{pmatrix}, \begin{pmatrix}1\\ -3\\ 1\end{pmatrix}$ form a basis for $\mathbb{R}^3$?

11. Use an LU factorization of $A=\begin{pmatrix}2 & 1 & 3\\ 4 & -1 & 3\\ -2 & 5 & 5\end{pmatrix}$ to solve $Ax=b$, where $b=\begin{pmatrix}1\\ -4\\ 9\end{pmatrix}$.

12. Find the numbers of a, b, c and d so that the matrix $\left[\begin{array}{ccc|c}1 & 2 & 3 & a\\ 0 & 4 & 5 & b\\ 0 & 0 & d & c\end{array}\right]$ has

(a) no solution (b) infinitely many solutions.

13. Which of the following vectors form a linearly independent set? Write one of the vectors as a linear combination of the others.

$$v_1=\begin{bmatrix}1\\ 4\\ 0\\ 3\end{bmatrix},\ v_2=\begin{bmatrix}1\\ 5\\ 3\\ -1\end{bmatrix},\ v_3=\begin{bmatrix}2\\ -1\\ 2\\ 6\end{bmatrix},\ v_4=\begin{bmatrix}-1\\ 4\\ -5\\ 1\end{bmatrix}$$

14. Describe the span of the vectors $\begin{bmatrix}-6\\ 7\\ 2\end{bmatrix},\begin{bmatrix}3\\ 2\\ 4\end{bmatrix},\begin{bmatrix}4\\ -1\\ 2\end{bmatrix}$.

Answer key

#1. (a) $P=\begin{pmatrix}4 & 2\\ 1 & 1\end{pmatrix}$, $D=\begin{pmatrix}0 & 0\\ 0 & -2\end{pmatrix}$. Please verify that $A=PDP^{-1}$. (b) $A^{25}=\begin{pmatrix}2^{25} & -2^{27}\\ 2^{24} & -2^{26}\end{pmatrix}$

#2. $\begin{pmatrix}2e^{-t}+4e^{6t}\\ -2e^{-t}+3e^{6t}\end{pmatrix}$

#3. Please do it yourself!

#4. (a) Yes (b) -8 (c) ± 1 (d) $\begin{pmatrix}0 & 1\\ 1 & 0\\ 1 & 1\end{pmatrix}$

#5. $\begin{pmatrix}1&0&0\\0&1&0\\3&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&2&1\end{pmatrix}\begin{pmatrix}1&2&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&3\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&-6\\0&1&0\\0&0&1\end{pmatrix}$ Note: It varies.

#6. $Q=\begin{pmatrix}1/5&-2/5&-4/5\\2/5&1/5&2/5\\2/5&-4/5&2/5\\4/5&2/5&-1/5\end{pmatrix}$ and $R=\begin{pmatrix}5&-2&1\\0&4&-1\\0&0&2\end{pmatrix}$ and least square solution is $\begin{pmatrix}-2/5\\0\\1\end{pmatrix}$.

#7. (a) $y=0.426x+0.826$ (b) about 5.9376 minutes

#8. $\lambda=0;\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}$ and $\lambda=-2;\begin{pmatrix}1\\-3\\-1\end{pmatrix}$

#9. $2\cdot\left(\frac{1}{2}\right)^{10}\begin{pmatrix}1\\-1\end{pmatrix}+3\cdot 2^{10}\begin{pmatrix}1\\1\end{pmatrix}$

#10. (a) $\begin{pmatrix}1&-2&0\\1&1&-3\end{pmatrix}$ (b) $\begin{pmatrix}9&-3/2&-5\\-5&1&3\\-2&1/2&1\end{pmatrix}$ (c) $col(A)=\{\begin{pmatrix}1\\2\\-3\\4\end{pmatrix},\begin{pmatrix}1\\-1\\2\\1\end{pmatrix},\begin{pmatrix}1\\1\\-2\\1\end{pmatrix}\}$, &

row(A)= $\{(1\quad 1\quad 3\quad 1),(2\quad -1\quad 0\quad 1),(4\quad 1\quad 6\quad 1)\}$

Null(A)= $row(A)^{\perp}=\{\begin{pmatrix}1\\2\\-1\\0\end{pmatrix}\}$ and $Null(A^T)=col(A)^{\perp}=\{\begin{pmatrix}1\\6\\3\\-1\end{pmatrix}\}$ (d) No

#11. L= $\begin{pmatrix}1&0&0\\2&1&0\\-1&-2&1\end{pmatrix}$ and U= $\begin{pmatrix}2&1&3\\0&-3&-3\\0&0&2\end{pmatrix}$ and $\vec{x}=\begin{pmatrix}1/2\\3\\-1\end{pmatrix}$

#12. (a) If d =0 and c =1 and a and b are arbitrary, then it has no solution. It varies.
(b) If all of a,b,c, and d are zero, then it has infinitely many solutions.

#13. The four vectors are linearly dependent and the last one is the linear combination of the first three $v_4=2v_1-v_2-v_3$.

#14. It is the plane given by $8x+10y-11z=0$.