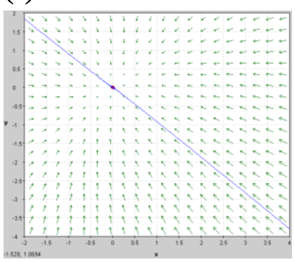
Review for the final
Math 286

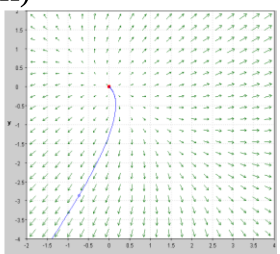
Final exam is 20% of the final grade so that it will affect your grade significantly. Please go over all necessary materials very carefully to do well on the final.

1. Match the phase plane, graph of x vs t according to a phase portrait and the system of differential equations. Write the type of the equilibrium solution and stability of it.
Show your work and give a reason for your choice.

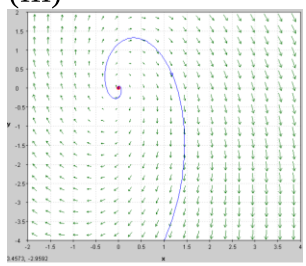
(a) $\begin{pmatrix} x_1' = 3x_1 + x_2 \\ x_2' = -13x_1 - 3x_2 \end{pmatrix}$ (b) $\begin{pmatrix} x_1' = x_1 + x_2 \\ x_2' = -4x_1 + x_2 \end{pmatrix}$ (c) $\begin{pmatrix} x_1' = 3x_1 + x_2 \\ x_2' = x_1 + 3x_2 \end{pmatrix}$ (d) $\begin{pmatrix} x_1' = -5x_1 + x_2 \\ x_2' = x_1 - 5x_2 \end{pmatrix}$

(I)

Type:

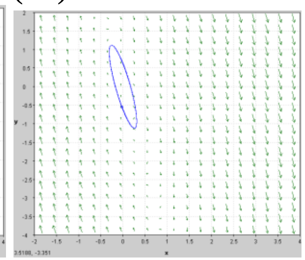
Stable or Unstable

(II)

Type:

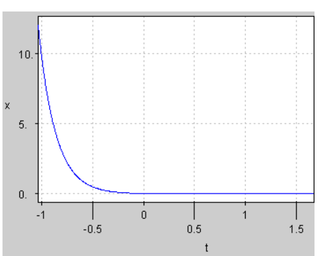
Stable or Unstable

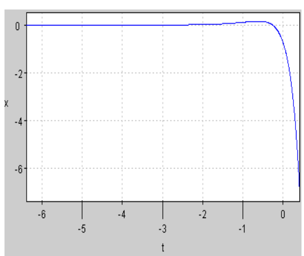
(III)

Type:

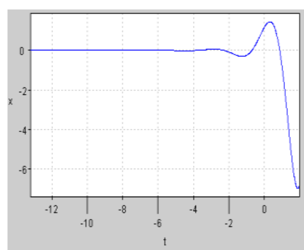
Stable or Unstable

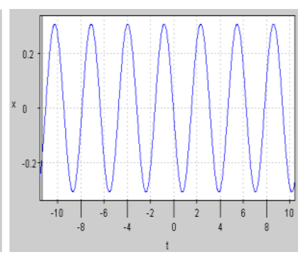
(IV)

Type:

Stable or Unstable

(i)


(ii)


(iii)


(iv)


2. Convert the second order DE $y'' - 3y' + 2y = 0$ into the system of first order DE.

3. Find the solution of the following IVP: $\begin{pmatrix} x_1' = x_1 + 4x_2 \\ x_2' = 4x_1 + 7x_2 \end{pmatrix}$, $x(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. What is the type of the equilibrium point?

4. A 500-gallon tank initially contains 100 gallons of brine in which 5 pounds of salt have been dissolved. Brine containing 2 pounds per gallon is added at the rate of 5 gallons per minute, and the mixture is poured out of the tank at the rate of 3 gallons per minute. Determine how much salt is in the tank at the moment it overflows.

5. Solve the following first order differential equation.

(a) $e^x \sin y - 2x + (e^x \cos y + 1)y' = 0$

(b) $xy' = \dfrac{y^2}{x} + y$ (c) $y' + \dfrac{1}{x} y = 3x^2 y^3$

(d) $(x^2 - 4)y' = y + 3$

6. Find the general solution of $y''-\frac{3}{x}y'+\frac{4}{x^2}y=0$ for x > 0, given that one solution is $y_1 = x^2$.

7. Solve the following second order differential equation.

(a) $y''+y'+6y=0$

(b) $x^2y''-5xy'+9y=0$

(c) $y''+2y'-3y=4e^{2x}$

8. If the object is released from rest from the equilibrium position, then the displacement function satisfies the IVP: $y''+6y'+5y=6\sqrt{5}\cos(\sqrt{5}t);\, y(0)=y'(0)=0$. Find the solution and explain the behavior of the solution as t increases.

9. Use the variation of parameters to find a general solution of $y''-\frac{4}{x}y'+\frac{4}{x^2}y=x^2+1$ for x > 0.

10. Solve the following using the Laplace transform.

(a) $y''+y=t;\, y(0)=1, y'(0)=0$

(b) $y''+y=g(t);\quad y(0)=0, y'(0)=1;\quad g(t)=\begin{cases} t/2, & 0\le t<6 \\ 3, & t\ge 6 \end{cases}$

11. Find the recurrence relation and use it to generate the first five terms of power series of the general solution. $y'+(1-x^2)y=x$

12. (a) Find the indicial equation. (b) From the roots of indicial equation to determine two linearly independent solutions for the following differential equation: $4xy''+2y'+y=0$

13. Draw the phase line and sketch several graphs of solutions in the ty-plane.

(a) $dy/dt = y(1-y)$

(b) $dy/dt = y^2(y-2)$

Answer key

1. (I) d-i Node, stable (II) c-ii Node, Unstable (III) b-iii Spiral, Unstable (IV) a-iv Center, stable

The solution of the system in #1 are given below.

(a) $x = c_1\begin{pmatrix} 3\cos 2t - 2\sin 2t \\ -13\cos 2t \end{pmatrix} + c_2\begin{pmatrix} 3\sin 2t + 2\cos 2t \\ -13\sin 2t \end{pmatrix}$ (b) $x = c_1 e^t\begin{pmatrix} \cos 2t \\ -2\sin 2t \end{pmatrix} + c_2 e^t\begin{pmatrix} \sin 2t \\ -2\cos 2t \end{pmatrix}$

(c) $x = c_1 e^{4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ (d) $x = c_1 e^{-6t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{-4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

2. $\begin{pmatrix} x_1' \\ x_1' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ i.e. $\begin{cases} x_1' = x_2 \\ x_2' = -2x_1 + 3x_2 \end{cases}$

3. $x = -\frac{4}{5}\begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-t} + \frac{7}{5}\begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{9t}$, saddle point, unstable

4. 982.56 pounds

5. (a) $e^x \sin y + y - x^2 = C$ (b) $y = -\frac{x}{\ln|x| + C}$ (c) $y = \sqrt{\frac{1}{-6x^3 + Cx^2}}$ (d) $(y+3)^4 = C(\frac{x-2}{x+2})$

6. $y = c_1 x^2 + c_2 x^2 \ln x$

7. (a) $y = c_1 e^{-x/2}\cos(\frac{\sqrt{23}x}{2}) + c_2 e^{-x/2}\sin(\frac{\sqrt{23}x}{2})$ (b) $y = c_1 x^3 + c_2 x^3 \ln x$ (c) $y = c_1 e^{-3x} + c_2 e^x + 4/5e^{2x}$

8. $y = \frac{\sqrt{5}}{4}(-e^{-t} + e^{-5t}) + \sin(\sqrt{5}t)$, exponential terms decrease to zero quite rapidly, giving less influence on the motion.

9. $y = c_1 x + c_2 x^4 - \frac{1}{9}x^4 - \frac{1}{2}x^2 + \frac{1}{3}x^4 \ln x$

10. (a) $f(t) = \cos t - \sin t + t$ (b) $y = \frac{1}{2}\sin t + \frac{1}{2}t - \frac{1}{2}u_6(t)[t - 6 - \sin(t-6)]$

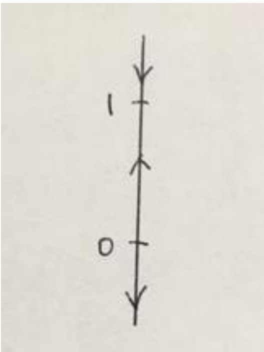
11. $y = a_0(1 - x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 - \frac{7}{4!}x^4 + ...) + [\frac{1}{2}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + ...]$

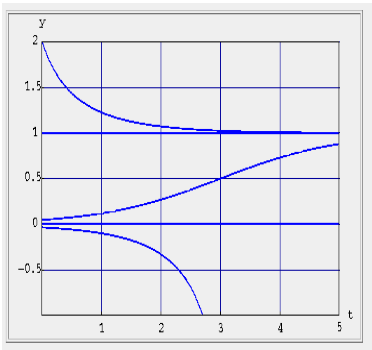
12. (a) $r_1 = 1/2, r_2 = 0$ (b) $y_1 = x^{1/2}[1 - \frac{1}{6}x + \frac{1}{120}x^2 - \frac{1}{5040}x^3 + ...] = x^{1/2}[1 + \sum_{n=1}^{\infty}\frac{(-1)^n}{2^n n![3\cdot 5\cdot 7\cdots(2n+1)]}x^n]$

$, y_2 = 1 - \frac{1}{2}x + \frac{1}{24}x^2 - \frac{1}{720}x^3 + ... = 1 + \sum_{n=1}^{\infty}\frac{(-1)^n}{2^n n![1\cdot 3\cdot 5\cdot 7\cdots(2n-1)]}x^n]$

13.

(a)





(b) leave it to you!