Math 408(77,134) Homework 12 Name & Number:

1. Show that a function f is analytic throughout the finite plane except for a finite number of points $z_1,\dots,z_n$. Show that

$$\operatorname*{Res}_{z=z_1} f(z)+\operatorname*{Res}_{z=z_2} f(z)+\dots+\operatorname*{Res}_{z=z_n} f(z)+\operatorname*{Res}_{z=\infty} f(z)=0.$$

2. Let the degrees of the polynomials

$$P(z)=a_0+a_1z+a_2z^2+\dots+a_nz^n,\quad (a_n\neq 0)$$

and $Q(z)=b_0+b_1z+b_2z^2+\dots+b_mz^m,\quad (b_m\neq 0)$ be such that $m\geq n+2$. Show that if all of the zeros of $Q(z)$ are interior to a simple closed contour C, then $\int_C \frac{P(z)}{Q(z)}\,dz=0$.

3. If $u(z)$ is harmonic for $|z|>R$ and continuous for $|z|\geq R$, show that for $\rho=Re^{i\phi},\ z=re^{i\theta}\ (r>R)$,

$$u(z)=-\frac{1}{2\pi}\int_0^{2\pi}\operatorname{Re}\left(\frac{\rho+z}{\rho-z}\right)u(Re^{i\phi})\,d\phi.$$

4. Let $F(\phi)$ be a continuous function of the real variable ϕ. Then the function $u(z)$ defined by

$$u(re^{i\theta})=\frac{1}{2\pi}\int_0^{2\pi}\frac{R^2-r^2}{R^2-2rR\cos(\theta-\phi)+r^2}F(\phi)\,d\phi,\quad (r<R).$$

Show that $u(z)$ is harmonic in the disk $|z|<R$.

5. Suppose $f(z)$ is analytic for $|z|<1$ with $f(0)=1$. If $\operatorname{Re} f(z)>0$ for $|z|<1$, then $|f(z)|\leq\frac{1+|z|}{1-|z|}$ for $|z|<1$.