Math 408(30-42) Homework 4 Name & Number:

1. (a) Use the expression $e^{iz} = \cos z + i\sin z$ to show that
$e^{iz_1}e^{iz_2} = \cos z_1 \cos z_2 - \sin z_1 \sin z_2 + i(\sin z_1 \cos z_2 + \cos z_1 \sin z_2)$
(b) Use the fact that $\cos(-z) = \cos z$ and $\sin(-z) = -\sin z$ to show that
$e^{-iz_1}e^{-iz_2} = \cos z_1 \cos z_2 - \sin z_1 \sin z_2 - i(\sin z_1 \cos z_2 + \cos z_1 \sin z_2)$
(c) Use (a) and (b) above to show that $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$.
(d) Show: $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$.

Note: (c) & (D) imply that $\sin 2z = 2\sin z \cos z$ and $\cos 2z = \cos^2 z - \sin^2 z$.

2. Find all the values of $\tan^{-1}(2i)$.

3. Show the following.
(a) $\tanh^{-1} z = \frac{1}{2}\log\frac{1+z}{1-z}$.
(b) $\frac{d}{dz}\sin^{-1} z = \frac{1}{(1-z^2)^{1/2}}$, $\frac{d}{dz}\cos^{-1} z = \frac{-1}{(1-z^2)^{1/2}}$ and $\frac{d}{dz}\tan^{-1} z = \frac{1}{1+z^2}$.
(c) $(1+i)^i = \exp(-\frac{\pi}{4} + 2n\pi)\exp(i\frac{\ln 2}{2})$ for $n = 0, \pm 1, \pm 2, \ldots$

4. Use derivative rules in Calculus to show the following rules for $w(t) = u(t) + iv(t)$ for a real variable t.
(a) $\frac{d}{dt}[z_0 w(t)] = z_0 w'(t)$, where z_0 is a complex constant.
(b) $\frac{d}{dt}[w(-t)] = -w'(-t)$, where $w'(-t)$ is the derivative of $w(t)$ with respect t evaluated at –t.

5. Find all roots of the equation $\log z = i\pi / 2$.

6. Show that $\mathrm{Log}(i^3) \neq 3\mathrm{Log}(i)$.

7. Write $|\exp(2z+i)|$ and $|\exp(iz^2)|$ in terms of x and y. Then show the following.

$$|\exp(2z+i) + \exp(iz^2)| \leq e^{2x} + e^{-2xy}$$

8. Show that $(z_1 z_2)^i \neq z_1^i z_2^i$ for $z_1 = 1 - i,\ z_2 = -1 - i$.