Math 408 Midterm Review

1. Solve $\cos z = 2$. Write your answer in Cartesian form in x+iy.

2. Use polar coordinates to compute $(1+i)^{100}$.

3. Sketch the set of points z in the complex plane determined by the condition below.
(a) $|z+i| = 2$
(b) $|z-1| \le |z+i|$

4. Show that the function $u(x,y) = \dfrac{y}{x^2+y^2}$ is harmonic in some domain and find the harmonic conjugate of u(x,y).

5. Determine where the function $f(z) = z^2 \operatorname{Im} z$ is analytic.

6. Compute the contour integrals along the curve given below.
(a) $\oint_C z-1\,dz$ along the semicircle $z = 1+e^{i\theta}$, $\pi \le \theta \le 2\pi$.
(b) $\oint_C z-1\,dz$ along the line from 1 and i.
(c) $\oint_C e^z\ dz$ along the curve C given by $r(t) = 1+i(t-1),\ 0 \le t \le 2$.
(d) $\dfrac{1}{2\pi i}\oint_{|z|=1} \dfrac{\sin z}{(z-z_0)^4}\ dz$ if $|z_0| < 1$.

7. Show that $(1+i)^i = \exp(-\dfrac{\pi}{4}+2n\pi)\exp(i\dfrac{\ln 2}{2}), n \in \mathbb{Z}$.

8. Show that the function $f(z) = 2xy + i(x^2-y^2)$ is nowhere analytic except at zero.

9. Let T be interior of the triangle with vertices 0,1,1+i. Find the image under the mapping $w = z^2$ and draw a picture.

10. Prove that there are two values of the constants $c \in \mathbb{R}$ such that $u(x,y) = e^{cy}\cos x$ is the real part of an analytic function. Find the analytic function $f(z)$ in each case.

11. (Prove or Disprove)
(a) If $w = f(z)$ is analytic in D, then $w = f(z)$ is continuous in D.
(b) The Mean Value theorem of derivatives is true for complex valued function.
(c) If a continuous function $w = f(z)$ has anti-derivative in a domain D, then $\int_{z_1}^{z_2} f(z)dz$ is path independent.
(d) There is no function that has two properties: $f(z)$ is analytic everywhere except the origin and $f(z)$ has derivative $f'(z) = \dfrac{1}{z}$ for $z \ne 0$.

12. Let C_R denote the upper half of the circle $|z| = R$ ($R > 2$), taken in the counterclockwise direction. Show that $\left|\oint_{C_R} \dfrac{2z^2-1}{z^4+5z^2+4}\,dz\right| \le \dfrac{\pi R(2R^2+1)}{(R^2-1)(R^2-4)}$. Then, show that the value of the integral tends to zero as R tends to infinity.