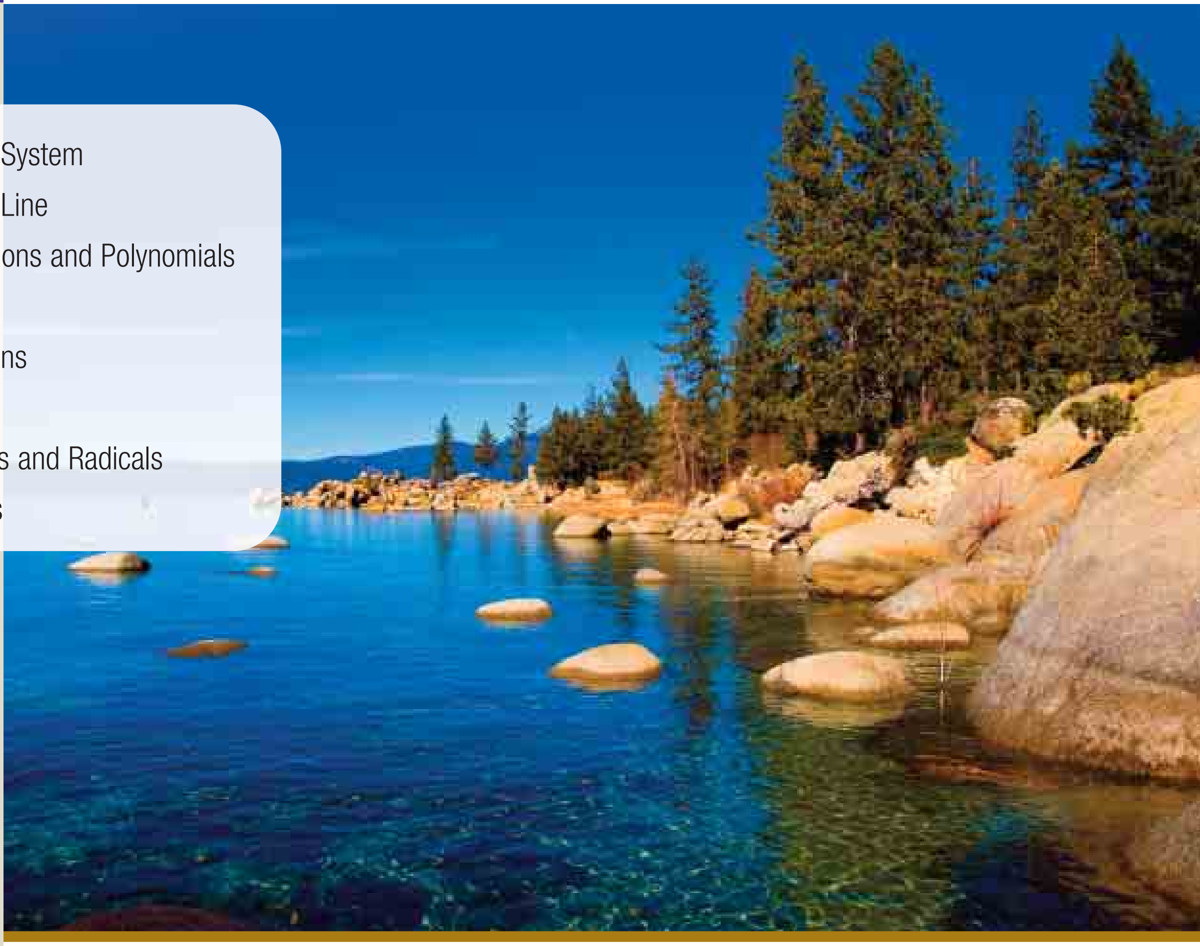
1 The Foundations of Algebra



Suppose you asked a friend of yours, who is a physics major, "How long does it take for a rock to reach the ground after being thrown into the air?" She will tell you that an object thrown straight up with a velocity of 20 meters per second would reach the ground in a little more than 4 seconds, if air resistance was not a factor. This is true, however, only on the Earth. What if we were on another planet, or even a large moon like Ganymede? An object thrown straight up from the surface of Ganymede, with the same initial velocity of 20 m/s, would take almost 20 seconds to reach the ground. (Check out the Chapter Project.)

If you asked your friend how she arrived at these conclusions, she could use words like ***algebraic expression, factoring,*** and ***polynomial***. Before you read this chapter, explore one of these words at http://mathworld.wolfram.com/Polynomial.html. This site can help you discover the meanings of many other terms as well.

Many problems that each of us encounters in the real world require the use and understanding of mathematics. Often, the methods used to solve these problems share certain characteristics, and it is both helpful and important to focus on these similarities. Algebra is one branch of mathematics that enables us to learn basic problem-solving techniques applicable to a wide variety of circumstances.

For example, if one starts with 2 apples and gets 3 more apples, how many apples does one have? If the travel time between Philadelphia and New York was 2 hours in the morning and 3 hours in the afternoon, how much time was spent traveling? The solution to the first problem is

$$2 \text{ apples} + 3 \text{ apples} = 5 \text{ apples}$$

The solution to the second problem is

$$2 \text{ hours} + 3 \text{ hours} = 5 \text{ hours}$$

Algebra focuses on the fact that

$$2x + 3x = 5x$$

It does not matter what meaning one gives to the symbol x.

Although this level of abstraction can create some difficulty, it is the nature of algebra that permits us to distill the essentials of problem solving into such rudimentary formulas. In the examples noted above, we used the *counting or natural numbers* as the number system needed to describe the problems. This number system is generally the first that one learns as a child. One can create other formulas for more general number systems.

We shall begin our presentation with a discussion of the *real number system* and its associated properties. We note a correspondence between the real numbers and the points on a real number line, and give a graphical presentation of this correspondence. The remainder of this chapter is devoted to a review of some fundamentals of algebra: the meaning and use of variables, algebraic expressions and polynomial forms, scientific notation, factoring, operations with algebraic fractions, and an introduction to the *complex number system*.

1.1 The Real Number System

Sets

We will need to use the notation and terminology of sets from time to time. A set is simply a collection of objects or numbers that are called the **elements** or **members** of the set. The elements of a set are written within braces so that the notation

$$A = \{4, 5, 6\}$$

tells us that the set A consists of the numbers 4, 5, and 6. The set

$$B = \{\text{Exxon, Ford, Sony}\}$$

consists of the names of these three corporations. We also write $4 \in A$, which we read as "4 is a member of the set A." Similarly, Ford $\in B$ is read as "Ford is a member of the set B," and Chrysler $\notin B$ is read as "Chrysler is not a member of the set B."

If every element of a set A is also a member of a set B, then A is a subset of B. For example, the set of all robins is a subset of the set of all birds.

EXAMPLE 1 SET NOTATION AND PROPERTIES

The set C consists of the names of all coins whose denominations are less than 50 cents. We may write C in set notation as follows:

$$C = \{\text{penny, nickel, dime, quarter}\}$$

We see that dime $\in C$, but half dollar $\notin C$. Further, the set $H = \{\text{nickel, dime}\}$ is a subset of C.

✔ Progress Check

The set V consists of the vowels in this particular sentence.

a. Write V in set notation.
b. Is the letter k a member of V?
c. Is the letter u a member of V?
d. List the subsets of V having four elements.

Answers

a. $V = \{a, e, i, o, u\}$ b. No c. Yes
d. $\{a, e, i, o\}, \{e, i, o, u\}, \{a, i, o, u\}, \{a, e, o, u\}, \{a, e, i, u\}$

The Set of Real Numbers

Since much of our work in algebra deals with the real numbers, we begin with a review of the composition of these numbers.

The numbers 1, 2, 3, . . . , used for counting, form the set of **natural numbers**. If we had only these numbers to use to show the profit earned by a company, we would have no way to indicate that the company had no profit or had a loss. To indicate no profit we introduce 0, and for losses we need to introduce negative numbers. The numbers

$$\ldots, -2, -1, 0, 1, 2, \ldots$$

form the set of **integers**. Thus, every natural number is an integer, and the set of natural numbers is seen to be a subset of the set of integers.

When we try to divide two apples equally among four people, we find no number in the set of integers that expresses how many apples each person should get. We need to introduce the set of **rational numbers**, which are numbers that can be written as a ratio of two integers.

$$\frac{p}{q} \text{ with } q \text{ not equal to zero}$$

Examples of rational numbers are

$$0 \quad \frac{2}{3} \quad -4 \quad \frac{7}{5} \quad \frac{-3}{4}$$

By writing an integer n in the form $\frac{n}{1}$, we see that every integer is a rational number. The decimal number 1.3 is also a rational number since $1.3 = \frac{13}{10}$.

We have now seen three fundamental sets of numbers: the set of natural numbers, the set of integers, and the set of rational numbers. Each successive set includes the previous set or sets, and each is more complicated than the one before. However, the set of rational numbers is still inadequate for sophisticated uses of mathematics, since there exist numbers that are not rational, that is, numbers that cannot be written as the ratio of two integers. These are called **irrational numbers**. It can be shown that the number a that satisfies $a \cdot a = 2$ is such a number. The number π, which is the ratio of the circumference of a circle to its diameter, is also such a number.

The decimal form of a rational number always forms a repeating pattern, such as

$$\begin{aligned} \frac{1}{2} &= 0.5000\ldots \\ \frac{13}{10} &= 1.3000\ldots \\ \frac{1}{3} &= 0.333\ldots \\ \frac{-2}{11} &= -0.\underbrace{18}\underbrace{18}\underbrace{18}\ldots \end{aligned}$$

(*Note*: The three dots, known as ellipses, following the numbers in each of the examples above means that the pattern continues in the same manner forever.)

The decimal form of an irrational number never forms a repeating pattern. The rational and irrational numbers together form the set of **real numbers**. (See Figure 1.)

FIGURE 1 The Set of Real Numbers

Calculator Alert

(1) Rational Numbers

A calculator display shows only a finite number of digits, which is often an approximation of the exact answer. Use your calculator to convert the rational numbers $\frac{1}{2}$, $\frac{13}{10}$, $\frac{1}{3}$, and $\frac{-2}{11}$ to decimal form, and note how these representations differ from those shown above.

$$\begin{aligned} 1 \div 2 &= 0.5 \\ 13 \div 10 &= 1.3 \\ 1 \div 3 &= 0.3333333333 \\ -2 \div 11 &= -0.1818181818 \end{aligned}$$

Which representations are exact and which are approximate?

(2) Irrational Numbers

Most calculators provide a rational decimal *approximation* to irrational numbers. For example, ten-digit approximations to $\sqrt{2}$ and π are

$$\begin{aligned} \sqrt{2} &= 1.414213562 \\ \pi &= 3.141592654 \end{aligned}$$

The System of Real Numbers

The system of real numbers consists of the set of real numbers together with the operations of addition and multiplication; in addition, this system satisfies the properties listed in Table 1, where a, b, and c denote real numbers.

EXAMPLE 2 PROPERTIES OF REAL NUMBERS

Specify the property in Table 1 illustrated by each of the following statements.

a. $2 + 3 = 3 + 2$

b. $(2 \cdot 3) \cdot 4 = 2 \cdot (3 \cdot 4)$

c. $2 \cdot \dfrac{1}{2} = 1$

d. $2(3 + 5) = (2 \cdot 3) + (2 \cdot 5)$

SOLUTION

a. commutative under addition
b. associative under multiplication
c. multiplicative inverse
d. distributive law

Equality

When we say that two numbers are **equal**, we mean that they represent the same value. Thus, when we write

$$a = b$$

Read "a equals b", we mean that a and b represent the same number. For example, $2 + 5$ and $4 + 3$ are different ways of writing the number 7, so we can write

$$2 + 5 = 4 + 3$$

Equality satisfies four basic properties shown in Table 2, where a, b, and c are any real numbers.

TABLE 1 Properties of Real Numbers

Example	Algebraic Expression	Property
$3 + 4$ is a real number.	$a + b$ is a real number.	**Closure under addition** The sum of two real numbers is a real number.
$2 \cdot 5$ is a real number.	$a \cdot b$ is a real number.	**Closure under multiplication** The product of two real numbers is a real number.
$4 + 8 = 8 + 4$	$a + b = b + a$	**Commutative under addition** We may add real numbers in any order.
$3(5) = 5(3)$	$a(b) = b(a)$	**Commutative under multiplication** We may multiply real numbers in any order.
$(2 + 5) + 3 = 2 + (5 + 3)$	$(a + b) + c = a + (b + c)$	**Associative under addition** We may group the addition of real numbers in any order.
$(2 \cdot 5)3 = 2(5 \cdot 3)$	$(ab)c = a(bc)$	**Associative under multiplication** We may group the multiplication of real numbers in any order.
$4 + 0 = 4$	$a + 0 = a$	**Additive identity** The sum of the unique real number 0 and any real number leaves that number unchanged.
$3(1) = 3$	$a(1) = a$	**Multiplicative identity** The product of the unique real number 1 and any real number leaves that number unchanged.
$5 + (-5) = 0$	$a + (-a) = 0$	**Additive inverse** The number $-a$ is called the **negative, opposite,** or **additive inverse** of a. If $-a$ is added to a, the result is the additive identity 0.

TABLE 1 Properties of Real Numbers (cont.)

Example	Algebraic Expression	Property
$7\left(\frac{1}{7}\right) = 1$	If $a \neq 0$, $a\left(\frac{1}{a}\right) = 1$	**Multiplicative inverse** The number $\frac{1}{a}$ is called the **reciprocal**, or **multiplicative inverse**, of a. If $\frac{1}{a}$ is multiplied by a, the result is the multiplicative identity 1.
$2(5 + 3) = (2 \cdot 5) + (2 \cdot 3)$ $(4 + 7)2 = (4 \cdot 2) + (7 \cdot 2)$	$a(b + c) = ab + ac$ $(a + b)c = ac + bc$	**Distributive laws** If one number multiplies the sum of two numbers, we may add the two numbers first and then perform the multiplication; or we may multiply each pair and then add the two products.

EXAMPLE 3 PROPERTIES OF EQUALITY

Specify the property in Table 2 illustrated by each of the following statements.

a. If $5a - 2 = b$, then $b = 5a - 2$.

b. If $a = b$ and $b = 5$, then $a = 5$.

c. If $a = b$, then $3a + 6 = 3b + 6$.

SOLUTION

a. symmetric property b. transitive property c. substitution property

Additional Properties

Using the properties of real numbers, the properties of equalities, and rules of logic, we can derive many other properties of the real numbers, as shown in Table 3, where a, b, and c are any real numbers.

TABLE 2 Properties of Equality

Example	Algebraic Expression	Property
$3 = 3$	$a = a$	Reflexive property
If $\frac{6}{3} = 2$ then $2 = \frac{6}{3}$.	If $a = b$ then $b = a$.	Symmetric property
If $\frac{6}{3} = 2$ and $2 = \frac{8}{4}$, then $\frac{6}{3} = \frac{8}{4}$	If $a = b$ and $b = c$, then $a = c$.	Transitive property
If $\frac{6}{3} = 2$, then we may replace $\frac{6}{3}$ by 2 or we may replace 2 by $\frac{6}{3}$.	If $a = b$, then we may replace a by b or we may replace b by a.	Substitution property

TABLE 3 Additional Properties of Real Numbers

Example	Algebraic Expression	Property
If $\frac{6}{3} = 2$ then $\frac{6}{3} + 4 = 2 + 4$ $\frac{6}{3}(5) = 2(5)$	If $a = b$, then $a + c = b + c$ $ac = bc$	The same number may be added to both sides of an equation. Both sides of an equation may be multiplied by the same number.
If $\frac{6}{3} + 4 = 2 + 4$ then $\frac{6}{3} = 2$.	If $a + c = b + c$ then $a = b$.	Cancellation law of addition
If $\frac{6}{3}(5) = 2(5)$ then $\frac{6}{3} = 2$.	If $ac = bc$ with $c \neq 0$ then $a = b$.	Cancellation law of multiplication
$2(0) = 0(2) = 0$ $2(3) = 0$ is impossible.	$a(0) = 0(a) = 0$ If $ab = 0$ then $a = 0$ or $b = 0$.	The product of two real numbers can be zero only if one of them is zero. The real numbers a and b are said to be **factors** of the product ab.
$-(-3) = 3$	$-(-a) = a$	Rules of signs
$(-2)(3) = (2)(-3) = -6$	$(-a)(b) = (a)(-b) = -(ab)$	
$(-1)(3) = -3$	$(-1)(a) = -a$	
$(-2)(-3) = 6$	$(-a)(-b) = ab$	
$(-2) + (-3) = -(2 + 3) = -5$	$(-a) + (-b) = -(a + b)$	

We next introduce the operations of subtraction and division. If a and b are real numbers, the *difference* between a and b, denoted by $a - b$, is defined by

$$a - b = a + (-b)$$

and the operation is called *subtraction*. Thus,

$$6 - 2 = 6 + (-2) = 4 \qquad 2 - 2 = 0 \qquad 0 - 8 = -8$$

We can show that the distributive laws hold for subtraction, that is,

$$a(b - c) = ab - ac$$

$$(a - b)c = ac - bc$$

Calculator Alert

In addition to a key for subtraction, most calculators have a key to represent negative numbers. This key may be labeled $\boxed{+/-}$, $\boxed{(-)}$, or $\boxed{\text{CHS}}$. We write -6, for example, to represent the keystrokes necessary to enter a negative number into your calculator. You must select the keystrokes that are appropriate for your calculator.

If a and b are real numbers and $b \neq 0$, then the *quotient* of a and b, denoted $\frac{a}{b}$ or a/b, is defined by

$$\frac{a}{b} = a \cdot \frac{1}{b}$$

and the operation is called *division*. We also write $\frac{a}{b}$ as $a \div b$ and speak of the *fraction* a over b. The numbers a and b are called the *numerator* and *denominator* of the fraction $\frac{a}{b}$, respectively. Observe that we have not defined division by zero, since 0 has no reciprocal.

In Table 4, a, b, c, and d are real numbers with $b \neq 0$, $c \neq 0$, and $d \neq 0$.

✔ Progress Check

Perform the indicated operations.

a. $\dfrac{3}{5} + \dfrac{1}{4}$ b. $\dfrac{5}{2} \cdot \dfrac{4}{15}$ c. $\dfrac{2}{3} + \dfrac{3}{7}$

Answers

a. $\dfrac{17}{20}$ b. $\dfrac{2}{3}$ c. $\dfrac{23}{21}$

TABLE 4 Additional Properties of Real Numbers

Example	Algebraic Expression	Property
$\dfrac{6}{10} = \dfrac{2 \cdot 3}{2 \cdot 5} = \dfrac{3}{5}$	$\dfrac{ac}{bc} = \dfrac{a}{b}$	Rules of fractions
$\dfrac{2}{3} \cdot \dfrac{5}{7} = \dfrac{10}{21}$	$\dfrac{a}{b} \cdot \dfrac{c}{d} = \dfrac{ac}{bd}$	
$\dfrac{4}{6} = \dfrac{2}{3}$ since $4 \cdot 3 = 6 \cdot 2$	$\dfrac{a}{b} = \dfrac{c}{d}$ if $ad = bc$	
$\dfrac{2}{9} + \dfrac{5}{9} = \dfrac{7}{9}$	$\dfrac{a}{d} + \dfrac{c}{d} = \dfrac{a + c}{d}$	
$\dfrac{\frac{2}{9}}{\frac{5}{9}} = \dfrac{\frac{2}{9}}{\frac{5}{9}} \cdot \dfrac{(9)}{(9)} = \dfrac{2}{5}$	$\dfrac{\frac{a}{d}}{\frac{b}{d}} = \dfrac{\frac{a}{d}}{\frac{b}{d}} \cdot \dfrac{(d)}{(d)} = \dfrac{a}{b}$	
$\dfrac{2}{3} + \dfrac{5}{4} = \dfrac{2}{3} \cdot \dfrac{(4)}{(4)} + \dfrac{5}{4} \cdot \dfrac{(3)}{(3)}$ $= \dfrac{8}{12} + \dfrac{15}{12} = \dfrac{23}{12}$	$\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{a}{b} \cdot \dfrac{(d)}{(d)} + \dfrac{c}{d} \cdot \dfrac{(b)}{(b)}$ $= \dfrac{ad + cb}{bd}$	
$\dfrac{\frac{2}{3}}{\frac{5}{7}} = \dfrac{\frac{2}{3}}{\frac{5}{7}} \cdot \dfrac{(3 \cdot 7)}{(3 \cdot 7)} = \dfrac{2(7)}{5(3)} = \dfrac{14}{15}$	$\dfrac{\frac{a}{b}}{\frac{c}{d}} = \dfrac{\frac{a}{b}}{\frac{c}{d}} \cdot \dfrac{(bd)}{(bd)} = \dfrac{ad}{cb}$	

Exercise Set 1.1

In Exercises 1–8, write each set by listing its elements within braces.

1. The set of natural numbers from 3 to 7, inclusive
2. The set of integers between -4 and 2
3. The set of integers between -10 and -8
4. The set of natural numbers from -9 to 3, inclusive
5. The subset of the set $S = \{-3, -2, -1, 0, 1, 2\}$ consisting of the positive integers in S
6. The subset of the set $S = \{-\frac{2}{3}, -1.1, 3.7, 4.8\}$ consisting of the negative rational numbers in S
7. The subset of all $x \in S$, $S = \{1, 3, 6, 7, 10\}$, such that x is an odd integer
8. The subset of all $x \in S$, $S = \{2, 5, 8, 9, 10\}$ such that x is an even integer

In Exercises 9–22, determine whether the given statement is true (T) or false (F).

9. -14 is a natural number.
10. $-\frac{4}{5}$ is a rational number.
11. $\frac{\pi}{3}$ is a rational number.
12. $\frac{1.75}{18.6}$ is an irrational number.
13. -1207 is an integer.
14. 0.75 is an irrational number.
15. $\frac{4}{5}$ is a real number.
16. 3 is a rational number.
17. 2π is a real number.
18. The sum of two rational numbers is always a rational number.
19. The sum of two irrational numbers is always an irrational number.
20. The product of two rational numbers is always a rational number.
21. The product of two irrational numbers is always an irrational number.
22. The difference of two irrational numbers is always an irrational number.

In Exercises 23–36, the letters represent real numbers. Identify the property or properties of real numbers that justify each statement.

23. $a + x = x + a$
24. $(xy)z = x(yz)$
25. $xyz + xy = xy(z + 1)$
26. $x + y$ is a real number
27. $(a + b) + 3 = a + (b + 3)$
28. $5 + (x + y) = (x + y) + 5$
29. cx is a real number.
30. $(a + 5) + b = (a + b) + 5$
31. $uv = vu$
32. $x + 0 = x$
33. $a(bc) = c(ab)$
34. $xy - xy = 0$
35. $5 \cdot \dfrac{1}{5} = 1$
36. $xy \cdot 1 = xy$

In Exercises 37–40, find a counterexample; that is, find real values for which the statement is false.

37. $a - b = b - a$
38. $\dfrac{a}{b} = \dfrac{b}{a}$
39. $a(b + c) = ab + c$
40. $(a + b)(c + d) = ac + bd$

In Exercises 41–44, indicate the property or properties of equality that justify the statement.

41. If $3x = 5$, then $5 = 3x$.
42. If $x + y = 7$ and $y = 5$, then $x + 5 = 7$.
43. If $2y = z$ and $z = x + 2$, then $2y = x + 2$.
44. If $x + 2y + 3z = r + s$ and $r = x + 1$, then $x + 2y + 3z = x + 1 + s$.

In Exercises 45–49, a, b, and c are real numbers. Use the properties of real numbers and the properties of equality to prove each statement.

45. If $-a = -b$, then $ac = bc$.

46. If $a = b$ and $c \neq 0$, then $\frac{a}{c} = \frac{b}{c}$.

47. If $a - c = b - c$, then $a = b$.

48. $a(b - c) = ab - ac$

49. Prove that the real number 0 does not have a reciprocal. (*Hint:* Assume $b = \frac{1}{0}$ is the reciprocal of 0.) Supply a reason for each of the following steps.

$$\begin{aligned} 1 &= 0 \cdot \frac{1}{0} \\ &= 0 \cdot b \\ &= 0 \end{aligned}$$

Since this conclusion is impossible, the original assumption must be false.

50. Give three examples for each of the following:
   a. a real number that is not a rational number
   b. a rational number that is not an integer
   c. an integer that is not a natural number

51. Give three examples for each of the following:
   a. two rational numbers that are not integers whose sum is an integer
   b. two irrational numbers whose sum is a rational number

52. Find a subset of the reals that is closed with respect to addition and multiplication but not with respect to subtraction and division.

53. Perform the indicated operations. Verify your answers using your calculator.
   a. $(-8) + 13$
   b. $(-8) + (-13)$
   c. $8 - (-13)$
   d. $(-5)(3) - (-12)$
   e. $\left(\frac{8}{9} + 3\right) + \left(\frac{-5}{9}\right)$
   f. $\frac{-5}{\frac{3}{2}}$
   g. $\frac{\frac{5}{8}}{\frac{1}{2}}$
   h. $\frac{\frac{-2}{3}}{\frac{-4}{3}}$
   i. $\left(\frac{3}{4}\right)\left(\frac{21}{37}\right) + \left(\frac{3}{4}\right)\left(\frac{16}{37}\right)$
   j. $\dfrac{\frac{1}{3} - \left(\frac{-1}{4}\right)}{\frac{7}{8} - \frac{3}{16}}$
   k. $\dfrac{\left(\frac{3}{5}\right)\left(\frac{1}{7}\right)}{\frac{1}{2} + \frac{1}{3}}$
   l. $\frac{2}{5}\left(\frac{3}{2} \cdot \frac{4}{7}\right)$

54. What is the meaning attached to each of the following?
   a. $\frac{6}{0}$
   b. $\frac{0}{6}$
   c. $\frac{6}{6}$
   d. $\frac{0}{\frac{1}{2}}$
   e. $\frac{0}{0}$

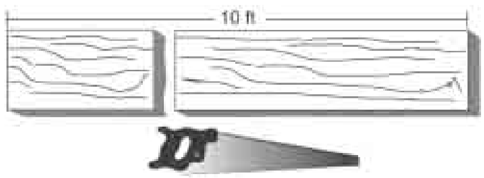
55. Use your calculator to convert the following fractions to (repeating) decimals. Look for a pattern that repeats.
   a. $\frac{1}{4}$ b. $-\frac{3}{5}$ c. $\frac{10}{13}$ d. $\frac{2}{7}$
   e. Does your calculator round off the final digit of an approximation, or does your calculator "drop off" the extra digits? To answer this question, evaluate $2 \div 3$ to see if your calculator displays 0.6666666666 or 0.6666666667.

56. A proportion is a statement of equality between two ratios. Solve the following proportions for x.
   a. $\frac{7}{8} = \frac{x}{12}$
   b. $\frac{7}{x} = \frac{11}{3}$

57. On a map of Pennsylvania, 1 inch represents 10 miles. Find the distance represented by 3.5 inches.

58. A car travels 135 miles on 6 gallons of gasoline. How far can it travel on 10 gallons of gasoline?

59. A board 10 feet long is cut into two pieces, the lengths of which are in the ratio of 2:3. Find the lengths of the pieces.



60. An alloy is $\frac{3}{8}$ copper, $\frac{5}{12}$ zinc, and the balance lead. How much lead is there in 282 pounds of alloy?

61. Which is the better value: 1 pound 3 ounces of beans for 85 cents or 13 ounces for 56 cents?

62. A piece of property is valued at \$28,500. What is the real estate tax at \$75.30 per \$1000.00 evaluation?

63. A woman's take-home pay is \$210.00 after deducting 18% withholding tax. What is her pay before the deduction?

64. List the set of possible ways of getting a total of 7 when tossing two standard dice.

65. A college student sent a postcard home with the following message:

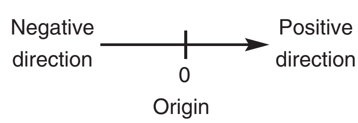
$$\begin{array}{r} \text{SEND} \\ \text{MORE} \\ \hline \text{MONEY} \end{array}$$

If each letter represents a different digit, and the calculation represents a sum, how much money did the student request?

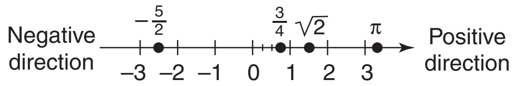
66. Eric starts at a certain time driving his car from New York to Philadelphia going 50 mph. Sixty minutes later, Steve leaves in his car en route from Philadelphia to New York going 40 mph. When the two cars meet, which one is nearer to New York?

1.2 The Real Number Line

There is a simple and very useful geometric interpretation of the real number system. Draw a horizontal line. Pick a point on this line, label it with the number 0, and call it the **origin**. Designate the side to the right of the origin as the *positive direction* and the side to the left as the *negative direction*.



Next, select a unit for measuring distance. With each positive integer n, we associate the point that is n units to the right of the origin. With each negative number $-n$, we associate the point that is n units to the left of the origin. Rational numbers, such as $\frac{3}{4}$ and $-\frac{5}{2}$, are associated with the corresponding points by dividing the intervals between integers into equal subintervals. Irrational numbers, such as $\sqrt{2}$ and π, can be written in decimal form. The corresponding points can be found by approximating these decimal forms to any desired degree of accuracy. Thus, the set of real numbers is identified with all possible points on this line. There is a real number for every point on the line; there is a point on the line for every real number. The line is called the **real number line**, and the number associated with a point is called its *coordinate*. We can now show some points on this line.



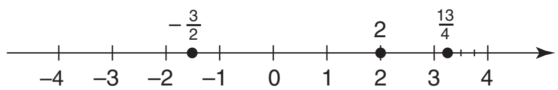
The numbers to the right of zero are called *positive*; the numbers to the left of zero are called *negative*. The positive numbers and zero together are called the **nonnegative numbers**.

We will frequently use the real number line to help picture the results of algebraic computations. For this purpose, we are only concerned with relative locations on the line. For example, it is adequate to show π slightly to the right of 3 since π is approximately 3.14.

EXAMPLE 1 REAL NUMBER LINE

Draw a real number line and plot the following points: $-\frac{3}{2}$, 2, $\frac{13}{4}$.

SOLUTION



Inequalities

If a and b are real numbers, we can compare their positions on the real number line by using the relations *less than*, *greater than*, *less than or equal to*, and *greater than or equal to*, as shown in Table 5.

TABLE 5 Inequalities

Symbol	Meaning
$<$	Less than
$>$	Greater than
$\leq$	Less than or equal to
$\geq$	Greater than or equal to

Table 6 describes both algebraic and geometric interpretations of the inequality symbols, where a and b are real numbers.

Expressions involving inequality symbols, such as $a < b$ and $a \geq b$, are called **inequalities**. We often combine these expressions so that $a \leq b < c$ means both $a \leq b$ and $b < c$. (*Note*: $a < c$ is also true.) For example, $-5 \leq x < 2$ is equivalent to $-5 \leq x$ and $x < 2$. Equivalently, x is between -5 and 2, including -5 and excluding 2.

✔ Progress Check

Verify that the following inequalities are true by using either the "Equivalent Statement" or the "Geometric Statement" of Table 6.

a. $-1 < 3$ b. $2 \le 2$

c. $-2.7 < -1.2$ d. $-4 < -2 < 0$

e. $-\frac{7}{2} < \frac{7}{2} < 7$

TABLE 6 Inequalities

Algebraic Expression	Meaning	Equivalent Statement	Geometric Statement
$a > 0$	a is greater than 0.	a is positive.	a lies to the right of the origin.
$a < 0$	a is less than 0.	a is negative.	a lies to the left of the origin.
$a > b$	a is greater than b.	$a - b$ is positive.	a lies to the right of b.
$a < b$	a is less than b.	$a - b$ is negative.	a lies to the left of b.
$a \geq b$	a is greater than or equal to b.	$a - b$ is positive or zero.	a lies to the right of b or coincides with b.
$a \leq b$	a is less than or equal to b.	$a - b$ is negative or zero.	a lies to the left of b or coincides with b.

TABLE 7 Properties of Inequalities

Example	Algebraic Expression	Property
Either $2 < 3$, $2 > 3$, or $2 = 3$.	Either $a < b$, $a > b$, or $a = b$.	Trichotomy property
Since $2 < 3$ and $3 < 5$, then $2 < 5$.	If $a < b$ and $b < c$ then $a < c$.	Transitive property
Since $2 < 5$, then $2 + 4 < 5 + 4$ or $6 < 9$.	If $a < b$ then $a + c < b + c$.	The sense of an inequality is preserved if any constant is added to both sides.
Since $2 < 3$ and $4 > 0$, then $2(4) < 3(4)$ or $8 < 12$.	If $a < b$ and $c > 0$, then $ac < bc$.	The sense of an inequality is preserved if it is multiplied by a positive constant.
Since $2 < 3$ and $-4 < 0$, then $2(-4) > 3(-4)$ or $-8 > -12$.	If $a < b$ and $c < 0$, then $ac > bc$.	The sense of an inequality is reversed if it is multiplied by a negative constant.

The real numbers satisfy the properties of inequalities shown in Table 7, where a, b, and c are real numbers.

EXAMPLE 2 PROPERTIES OF INEQUALITIES

a. Since $-2 < 4$ and $4 < 5$, then $-2 < 5$.

b. Since $-2 < 5$, $-2 + 3 < 5 + 3$, or $1 < 8$.

c. Since $3 < 4$, $3 + (-5) < 4 + (-5)$, or $-2 < -1$.

d. Since $2 < 5$, $2(3) < 5(3)$, or $6 < 15$.

e. Since $-3 < 2$, $(-3)(-2) > 2(-2)$, or $6 > -4$.

Absolute Value

Suppose we are interested in the *distances* between the origin and the points labeled 4 and -4 on the real number line. Each of these points is four units from the origin; that is, the *distance is independent of the direction* and is nonnegative. (See Figure 2.) Furthermore, the distance between 4 and -4 is 8 units.

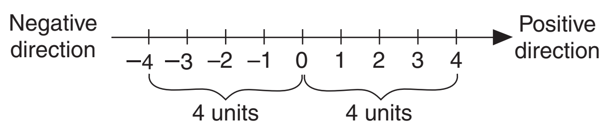


FIGURE 2 **Distance on the Real Number Line**

When we are interested in the magnitude of a number a, and do not care about the direction or sign, we use the concept of **absolute value**, which we write as $|a|$. The formal definition of absolute value is stated as follows:

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

Since distance is independent of direction and is always nonnegative, we can view $|a|$ as the distance from the origin to either point a or point $-a$ on the real number line.

EXAMPLE 3 ABSOLUTE VALUE AND DISTANCE

a. $|4| = 4$ $\quad |-4| = 4$ $\quad |0| = 0$

b. The distance on the real number line between the point labeled 3.4 and the origin is $|3.4| = 3.4$. Similarly, the distance between point -2.3 and the origin is $|-2.3| = 2.3$.

In working with the notation of absolute value, it is important to perform the operations within the bars first. Here are some examples.

EXAMPLE 4 ABSOLUTE VALUE

a. $|5-2| = |3| = 3$

b. $|2-5| = |-3| = 3$

c. $|3-5| - |8-6| = |-2| - |2| = 2-2 = 0$

d. $\dfrac{|4-7|}{-6} = \dfrac{|-3|}{-6} = \dfrac{3}{-6} = -\dfrac{1}{2}$

Graphing Calculator Alert

Your calculator may have an absolute value key, usually labeled $\boxed{\text{ABS}}$. If you have a graphing calculator, it is important to use parentheses when you use this key.

Examples:

a. *ABS(5 − 2)*

b. ABS(2 − 5)

c. ABS(3 − 5) − ABS(8 − 6)

d. ABS(4 − 7) ÷ (−6)

Table 8 describes the properties of absolute value where a and b are real numbers.

We began by showing a use for absolute value in denoting distance from the origin without regard to direction. We conclude by demonstrating the use of absolute value to denote the distance between *any* two points a and b on the real number line. In Figure 3, the distance between the points labeled 2 and 5 is 3 units and can be obtained by evaluating either $|5-2|$ or $|2-5|$. Similarly, the distance between the points labeled -1 and 4 is given by either $|4-(-1)| = 5$ or $|-1-4| = 5$.

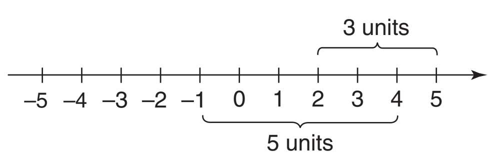


FIGURE 3 Distance on the Real Number Line

TABLE 8 Basic Properties of Absolute Value

Example	Algebraic Expression	Property
$\lvert -2 \rvert \geq 0$	$\lvert a \rvert \geq 0$	Absolute value is always nonnegative.
$\lvert 3 \rvert = \lvert -3 \rvert = 3$	$\lvert a \rvert = \lvert -a \rvert$	The absolute values of a number and its negative are the same.
$\lvert 2-5 \rvert = \lvert -3 \rvert = 3$ $\lvert 5-2 \rvert = \lvert 3 \rvert = 3$	$\lvert a-b \rvert = \lvert b-a \rvert$	The absolute value of the difference of two numbers is always the same, irrespective of the order of subtraction.
$\lvert (-2)(3) \rvert = \lvert -2 \rvert \lvert 3 \rvert = 6$	$\lvert ab \rvert = \lvert a \rvert \lvert b \rvert$	The absolute value of a product is the product of the absolute values.

Using the notation $\overline{AB}$ to denote the distance between the points A and B, we provide the following definition:

Distance on the Real Number Line

The **distance** $\overline{AB}$ between points A and B on the real number line, whose coordinates are a and b, respectively, is given by

$$\overline{AB} = \lvert b - a \rvert$$

The third property of absolute value from Table 8 tells us that $\overline{AB} = \lvert b - a \rvert = \lvert a - b \rvert$. Viewed another way, this property states that the distance between any two points on the real number line is independent of the direction.

EXAMPLE 5 DISTANCE ON THE REAL NUMBER LINE

Let points A, B, and C have coordinates -4, -1, and 3, respectively, on the real number line. Find the following distances.

a. $\overline{AB}$ b. $\overline{CB}$ c. $\overline{OB}$, where O is the origin

SOLUTION

Using the definition, we have

a. $\overline{AB} = \lvert -1 - (-4) \rvert = \lvert -1 + 4 \rvert = \lvert 3 \rvert = 3$

b. $\overline{CB} = \lvert -1 - 3 \rvert = \lvert -4 \rvert = 4$

c. $\overline{OB} = \lvert -1 - 0 \rvert = \lvert -1 \rvert = 1$

✔ Progress Check

The points P, Q, and R on the real number line have coordinates –6, 4, and 6, respectively. Find the following distances.

a. $\overline{PR}$ b. $\overline{QP}$ c. $\overline{PQ}$

Answers

a. 12 b. 10 c. 10

Exercise Set 1.2

1. Draw a real number line and plot the following points.

   a. 4 b. -2

   c. $\frac{5}{2}$ d. -3.5

   e. 0

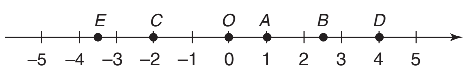
2. Draw a real number line and plot the following points.

   a. -5 b. 4

   c. -3.5 d. $\frac{7}{2}$

   e. -4

3. Give the real numbers associated with the points A, B, C, D, O, and E on the real number line below.



4. Represent the following by real numbers.

   a. a profit of \$10

   b. a loss of \$20

   c. a temperature of 20°F above zero

   d. a temperature of 5°F below zero

In Exercises 5–10, indicate which of the two given numbers appears first, viewed from left to right, on the real number line.

5. 4, 6
6. $\frac{1}{2}$, 0
7. $-2, \frac{3}{4}$
8. 0, -4
9. $-5, -\frac{2}{3}$
10. 4, -5

In Exercises 11–14, indicate the set of numbers on a real number line.

11. The natural numbers less than 8
12. The natural numbers greater than 4 and less than 10
13. The integers that are greater than 2 and less than 7
14. The integers that are greater than -5 and less than or equal to 1

In Exercises 15–24, express the statement as an inequality.

15. 10 is greater than 9.99.
16. -6 is less than -2.
17. a is nonnegative.
18. b is negative.
19. x is positive.
20. a is strictly between 3 and 7.
21. a is strictly between $\frac{1}{2}$ and $\frac{1}{4}$.
22. b is less than or equal to -4.
23. b is greater than or equal to 5.
24. x is negative.

In Exercises 25–30, give a property of inequalities that justifies the statement.

25. Since $-3 < 1$, then $-1 < 3$.
26. Since $-5 < -1$ and $-1 < 4$, then $-5 < 4$.

27. Since $14 > 9$, then $-14 < -9$.

28. Since $5 > 3$, then $5 \neq 3$.

29. Since $-1 < 6$, then $-3 < 18$.

30. Since $6 > -1$, then 7 is a positive number.

In Exercises 31–44, find the value of the expression. Verify your answer using your calculator.

31. $|2|$

32. $\left|-\frac{2}{3}\right|$

33. $|1.5|$

34. $|-0.8|$

35. $-|2|$

36. $-\left|-\frac{2}{5}\right|$

37. $|2 - 3|$

38. $|2 - 2|$

39. $|2 - (-2)|$

40. $|2| + |-3|$

41. $\dfrac{|14 - 8|}{|-3|}$

42. $\dfrac{|2 - 12|}{|1 - 6|}$

43. $\dfrac{|3| - |2|}{|3| + |2|}$

44. $\dfrac{|3 - 2|}{|3 + 2|}$

In Exercises 45–50, the coordinates of points A and B are given. Find $\overline{AB}$.

45. 2, 5

46. -3, 6

47. -3, -1

48. -4, $\frac{11}{2}$

49. $-\frac{4}{5}$, $\frac{4}{5}$

50. 2, 2

51. For what values of x and y is $|x + y| = |x| + |y|$?

52. For what values of x and y is $|x + y| < |x| + |y|$?

53. Find the set of integers whose distance from 3 is less than or equal to 5.

54. List the set of integers x such that
    a. $-2 < x < 3$
    b. $0 < x < 5$
    c. $-1 < 2x < 10$

55. For the inequality $-1 < 5$, state the resulting inequality when the following operations are performed on both sides.
    a. add 2
    b. subtract 5
    c. multiply by 2
    d. multiply by -5
    e. divide by -1
    f. divide by 2
    g. square

56. A computer sales representative receives \$400 monthly plus a 10% commission on sales. How much must she sell in a month for her income to be at least \$600 for that month?

In Exercises 57–62, use the coordinates given in Exercises 45–50 to find the midpoint of the interval.

63. For what values of x does each of the following hold?
    a. $|3 - x| = 3 - x$
    b. $|5x - 2| = -(5x - 2)$

64. Evaluate $\dfrac{|x - 3|}{x - 3}$ for $x = -2, -1, 0, 1, 2$. Make a conjecture about the value of this expression for all values of x.

1.3 Algebraic Expressions and Polynomials

A **variable** is a symbol to which we can assign values. For example, in Section 1.1 we defined a rational number as one that can be written as $\frac{p}{q}$, where p and q are integers and q is not zero. The symbols p and q are variables since we can assign values to them. A variable can be restricted to a particular number system (for example, p and q must be integers) or to a subset of a number system.

If we invest P dollars at an annual interest rate of 6%, then we will earn $0.06P$ dollars interest per year; and we will have $P + 0.06P$ dollars at the end of the year. We call $P + 0.06P$ an **algebraic expression**. Note that an algebraic expression involves **variables** (in this case P), **constants** (such as 0.06), and **algebraic operations** (such as $+$, $-$, $\times$, $\div$). Virtually everything we do in algebra involves algebraic expressions.

An algebraic expression takes on a **value** when we assign a specific number to each variable in the expression. Thus, the expression

$$\frac{3m + 4n}{m + n}$$

is **evaluated** when $m = 3$ and $n = 2$ by substitution of these values for m and n:

$$\frac{3(3) + 4(2)}{3 + 2} = \frac{9 + 8}{5} = \frac{17}{5}$$

We often need to write algebraic expressions in which a variable multiplies itself repeatedly. We use the notation of exponents to indicate such repeated multiplication. Thus,

$$a^1 = a \qquad a^2 = a \cdot a \qquad a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ factor}}$$

where n is a natural number and a is a real number. We call a the **base** and n the **exponent** and say that a^n is the nth **power** of a. When $n = 1$, we simply write a rather than a^1.

It is convenient to define a^0 for all real numbers $a \neq 0$ as $a^0 = 1$. We will provide motivation for this seemingly arbitrary definition in Section 1.7.

EXAMPLE 1 MULTIPLICATION WITH NATURAL NUMBER EXPONENTS

Write the following without using exponents.

a. $\left(\frac{1}{2}\right)^3$

b. $2x^3$

c. $(2x)^3$

d. $-3x^2y^3$

SOLUTION

a. $\left(\frac{1}{2}\right)^3 = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$

b. $2x^3 = 2 \cdot x \cdot x \cdot x$

c. $(2x)^3 = 2x \cdot 2x \cdot 2x = 8 \cdot x \cdot x \cdot x$

d. $-3x^2y^3 = -3 \cdot x \cdot x \cdot y \cdot y \cdot y$

WARNING

Note the difference between

$$(-3)^2 = (-3)(-3) = 9$$

and

$$-3^2 = -(3 \cdot 3) = -9$$

Calculator Alert

Your calculator evaluates exponents using a special key, which may be labeled $\boxed{x^y}$, $\boxed{y^x}$, or $\boxed{\wedge}$.

Example: $(1 \div 2)\boxed{x^y}3 = 0.125$

or $(1 \div 2)\boxed{y^x}3 = 0.125$

or $(1 \div 2)\boxed{\wedge}3 = 0.125$

We will use $\boxed{x^y}$ or $\boxed{\wedge}$ to indicate the exponentiation key in this text.

In addition to the exponentiation key, your calculator probably has a special key labeled $\boxed{x^2}$.

Examples: $(-3)\boxed{x^2} = 9$

$-3\boxed{x^2} = -9$

Later in this chapter we will need an important rule of exponents. Observe that if m and n are natural numbers and a is any real number, then

$$a^m \cdot a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{m \text{ factors}} \cdot \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ factors}}$$

Since there are a total of $m + n$ factors on the right side, we conclude that

$$a^m a^n = a^{m+n}$$

EXAMPLE 2 MULTIPLICATION WITH NATURAL NUMBER EXPONENTS

Multiply.

a. $x^2 \cdot x^3$

b. $(3x)(4x^4)$

SOLUTION

a. $x^2 \cdot x^3 = x^{2+3} = x^5$

b. $(3x)(4x^4) = 3 \cdot 4 \cdot x \cdot x^4 = 12x^{1+4} = 12x^5$

✔ Progress Check

Multiply.

a. $x^5 \cdot x^2$ b. $(2x^6)(-2x^4)$

Answers

a. x^7 b. $-4x^{10}$

Polynomials

A polynomial is an algebraic expression of a certain form. Polynomials play an important role in the study of algebra since many word problems translate into equations or inequalities that involve polynomials. We first study the manipulative and mechanical aspects of polynomials. This knowledge will serve as background for dealing with their applications in later chapters.

Let x denote a variable and let n be a constant, nonnegative integer. The expression ax^n, where a is a constant real number, is called a **monomial in x**. A **polynomial in x** is an expression that is a sum of monomials and has the general form

$$P = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \qquad a_n \neq 0 \tag{1}$$

Each of the monomials in Equation (1) is called a **term** of P, and $a_0, a_1, \ldots, a_n$ are constant real numbers that are called the **coefficients** of the terms of P. Note that a polynomial may consist of just one term; that is, a monomial is considered to be a polynomial.

EXAMPLE 3 POLYNOMIAL EXPRESSIONS

a. The following expressions are polynomials in x:

$$3x^4 + 2x + 5 \qquad 2x^3 + 5x^2 - 2x + 1 \qquad \frac{3}{2}x^3$$

Notice that we write $2x^3 + 5x^2 + (-2)x + 1$ as $2x^3 + 5x^2 - 2x + 1$.

b. The following expressions are not polynomials in x:

$$2x^{1/2} + 5 \qquad 3 - \frac{4}{x} \qquad \frac{2x-1}{x-2}$$

Remember that each term of a polynomial in x must be of the form ax^n, where a is a real number and n is a nonnegative integer.

The **degree of a monomial in *x*** is the exponent of x. Thus, the degree of $5x^3$ is 3. A monomial in which the exponent of x is 0 is called a **constant term** and is said to be of *degree zero*. The nonzero coefficient a_n of the term in P with highest degree is called the **leading coefficient** of P, and we say that P is a **polynomial of degree *n***. The polynomial whose coefficients are all zero is called the **zero polynomial**. It is denoted by 0 and is said to have no degree.

EXAMPLE 4 VOCABULARY OF POLYNOMIALS

Given the polynomial

$$P = 2x^4 - 3x^2 + \frac{4}{3}x - 1$$

The terms of P are

$$2x^4, \qquad 0x^3, \qquad -3x^2, \qquad \frac{4}{3}x, \qquad -1$$

The coefficients of the terms are

$$2, \qquad 0, \qquad -3, \qquad \frac{4}{3}, \qquad -1$$

The degree of P is 4 and the leading coefficient is 2.

A *monomial in the variables x and y* is an expression of the form ax^my^n, where a is a constant and m and n are constant, nonnegative integers. The number a is called the **coefficient** of the monomial. The *degree of a monomial in x and y* is the sum of the exponents of x and y. Thus, the degree of $2x^3y^2$ is $3 + 2 = 5$. A *polynomial in x and y* is an expression that is a sum of monomials. The *degree of a polynomial in x and y* is the degree of the highest degree monomial with nonzero coefficient.

EXAMPLE 5 DEGREE OF POLYNOMIALS

The following are polynomials in x and y:

$2x^2y + y^2 - 3xy + 1$	Degree is 3.
xy	Degree is 2.
$3x^4 + xy - y^2$	Degree is 4.

Operations with Polynomials

If P and Q are polynomials in x, then the terms ax^r in P and bx^r in Q are said to be **like terms**; that is, like terms have the same exponent in x. For example, given

$$P = 4x^2 + 4x - 1$$

and

$$Q = 3x^3 - 2x^2 + 4$$

then the like terms are $0x^3$ and $3x^3$, $4x^2$ and $-2x^2$, $4x$ and $0x$, -1 and 4.

We define equality of polynomials in the following way:

> Two polynomials are equal if all like terms are equal.

EXAMPLE 6 EQUALITY OF POLYNOMIALS

Find A, B, C, and D if

$$Ax^3 + (A + B)x^2 + Cx + (C - D) = -2x^3 + x + 3$$

SOLUTION

Equating the coefficients of the terms, we have

$$\begin{array}{llll} A = -2 \qquad & A + B = 0 \qquad & C = 1 \qquad & C - D = 3 \\ & \phantom{A + {}} B = 2 & & \phantom{C - {}} D = -2 \end{array}$$

If P and Q are polynomials in x, the *sum* $P + Q$ is obtained by forming the sums of all pairs of like terms. The sum of ax^r in P and bx^r in Q is $(a + b)x^r$. Similarly, the *difference* $P - Q$ is obtained by forming the differences, $(a - b)x^r$, of like terms.

EXAMPLE 7 ADDITION AND SUBTRACTION OF POLYNOMIALS

a. Add $2x^3 + 2x^2 - 3$ and $x^3 - x^2 + x + 2$.

b. Subtract $2x^3 + x^2 - x + 1$ from $3x^3 - 2x^2 + 2x$.

SOLUTION

a. Adding the coefficients of like terms,

$$(2x^3 + 2x^2 - 3) + (x^3 - x^2 + x + 2) = 3x^3 + x^2 + x - 1$$

b. Subtracting the coefficients of like terms,

$$(3x^3 - 2x^2 + 2x) - (2x^3 + x^2 - x + 1) = x^3 - 3x^2 + 3x - 1$$

WARNING

$$(x + 5) - (x + 2) \neq (x + 5) - x + 2$$

The coefficient -1 must multiply each term in the parentheses. Thus,

$$-(x + 2) = -x - 2$$

Therefore,

$$(x + 5) - (x + 2) = x + 5 - x - 2 = 3$$

while

$$(x + 5) - x + 2 = x + 5 - x + 2 = 7$$

Multiplication of polynomials is based on the rule for exponents developed earlier in this section,

$$a^m a^n = a^{m+n}$$

and on the distributive laws

$$a(b + c) = ab + ac$$
$$(a + b)c = ac + bc$$

EXAMPLE 8 MULTIPLICATION OF POLYNOMIALS

Multiply $3x^3(2x^3 - 6x^2 + 5)$.

SOLUTION

$3x^3(2x^3 - 6x^2 + 5)$

$= (3x^3)(2x^3) + (3x^3)(-6x^2) + (3x^3)(5)$ Distributive law

$= (3)(2)x^{3+3} + (3)(-6)x^{3+2} + (3)(5)x^3$ $a^m a^n = a^{m+n}$

$= 6x^6 - 18x^5 + 15x^3$

EXAMPLE 9 MULTIPLICATION OF POLYNOMIALS

Multiply $(x + 2)(3x^2 - x + 5)$.

SOLUTION

$(x + 2)(3x^2 - x + 5)$

$= x(3x^2 - x + 5) + 2(3x^2 - x + 5)$ Distributive law

$= 3x^3 - x^2 + 5x + 6x^2 - 2x + 10$ Distributive law and $a^m a^n = a^{m+n}$

$= 3x^3 + 5x^2 + 3x + 10$ Adding like terms

✔ Progress Check

Multiply.

a. $(x^2 + 2)(x^2 - 3x + 1)$ b. $(x^2 - 2xy + y)(2x + y)$

Answers

a. $x^4 - 3x^3 + 3x^2 - 6x + 2$ b. $2x^3 - 3x^2y + 2xy - 2xy^2 + y^2$

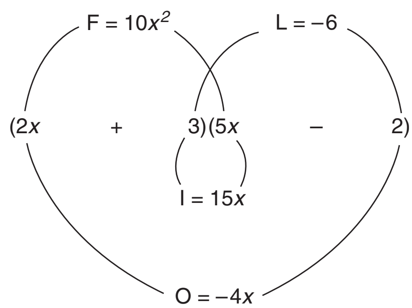
The multiplication in Example 9 can be carried out in "long form" as follows:

$$
\begin{array}{rl}
3x^2 - x + 5 & \\
x + 2 & \\
\hline
3x^3 - x^2 + 5x \phantom{{}+10} & = x(3x^2 - x + 5) \\
6x^2 - 2x + 10 & = 2(3x^2 - x + 5) \\
\hline
3x^3 + 5x^2 + 3x + 10 & = \text{sum of above lines}
\end{array}
$$

In Example 9, the product of polynomials of degrees 1 and 2 is seen to be a polynomial of degree 3. From the multiplication process, we can derive the following useful rule:

> The degree of the product of two nonzero polynomials is the sum of the degrees of the polynomials.

Products of the form $(2x + 3)(5x - 2)$ or $(2x + y)(3x - 2y)$ occur often, and we can handle them by the method sometimes referred to as FOIL: F = first, O = outer, I = inner, L = last.



$$
\begin{array}{rcl}
\textbf{F} & = & (2x)(5x) = 10x^2 \\
\textbf{O} & = & (2x)(-2) = -4x \\
\textbf{I} & = & (3)(5x) = 15x \\
\textbf{L} & = & (3)(-2) = -6 \\
\hline
\text{Sum} & = & 10x^2 - 4x + 15x - 6 \\
 & = & 10x^2 + 11x - 6
\end{array}
$$

A number of special products occur frequently, and it is worthwhile knowing them.

Special Products

$$(a + b)(a - b) = a^2 - b^2$$

$$(a + b)^2 = (a + b)(a + b) = a^2 + 2ab + b^2$$

$$(a - b)^2 = (a - b)(a - b) = a^2 - 2ab + b^2$$

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

EXAMPLE 10 MULTIPLICATION OF POLYNOMIALS

Multiply.

a. $(x + 2)^2$ b. $(x - 3)^2$ c. $(x + 4)(x - 4)$

SOLUTION

a. $(x + 2)^2 = (x + 2)(x + 2) = x^2 + 4x + 4$

b. $(x - 3)^2 = (x - 3)(x - 3) = x^2 - 6x + 9$

c. $(x + 4)(x - 4) = x^2 - 16$

✔ Progress Check

a. Multiply $(2x^2 - xy + y^2)(3x + y)$.   b. Multiply $(2x - 3)(3x - 2)$.

Answers

a. $6x^3 - x^2y + 2xy^2 + y^3$ b. $6x^2 - 13x + 6$

Graphing Calculator Power User's Corner

(1) Assigning Values to Variables

Assign values to variables on your graphing calculator using the STORE command. There is usually an arrow key $\boxed{\rightarrow}$ or a key labeled $\boxed{\text{STO}}$. For example, to set $M = 3$, you press

$$\boxed{3}\ \boxed{\rightarrow}\ \boxed{M} \qquad \text{or} \qquad \boxed{3}\ \boxed{\text{STO}}\ \boxed{M}$$

Check your owner's manual for details. The owner's manual may be available online. Look up your calculator by model and number.

(2) Evaluating Algebraic Expressions on a Graphing Calculator

Once specific values have been assigned to variables in your graphing calculator, you can use these variables to evaluate algebraic expressions. For example, evaluate

$$\frac{3m + 4n}{m + n}$$

when $m = 3$ and $n = 2$.

Step 1. Store 3 in memory location M and store 2 in memory location N.

Step 2. Enter and evaluate the expression

$$(3M + 4N) \div (M + N)$$

Graphing Calculator Power User's Corner (cont.)

Note that the numerator expression and the denominator expression must both be enclosed in parentheses. On some calculators, you may need to enter $3 \times M$ and $4 \times N$ to multiply.

Step 3. Note that your answer is given in the decimal form 3.4. Use your calculator to verify that $\frac{17}{5} = 3.4$.

Exercise Set 1.3

In Exercises 1–6, evaluate the given expression when $r = 2$, $s = -3$, and $t = 4$.

1. $r + 2s + t$
2. rst
3. $\dfrac{rst}{r + s + t}$
4. $(r + s)t$
5. $\dfrac{r + s}{rt}$
6. $\dfrac{r + s + t}{t}$
7. Evaluate $\frac{2}{3}r + 5$ when $r = 12$.
8. Evaluate $\frac{9}{5}C + 32$ when $C = 37$.
9. If P dollars are invested at a simple interest rate of r percent per year for t years, the amount on hand at the end of t years is $P + Prt$. Suppose you invest \$2000 at 8% per year ($r = 0.08$). Find the amount you will have on hand after

   a. 1 year b. $\frac{1}{2}$ year c. 8 months.
10. The perimeter of a rectangle is given by the formula $P = 2(L + W)$, where L is the length and W is the width of the rectangle. Find the perimeter if

    a. $L = 2$ feet, $W = 3$ feet

    b. $L = \frac{1}{2}$ meter, $W = \frac{1}{4}$ meter
11. Evaluate $0.02r + 0.314st + 2.25t$ when $r = 2.5$, $s = 3.4$, and $t = 2.81$.
12. Evaluate $10.421x + 0.821y + 2.34xyz$ when $x = 3.21$, $y = 2.42$, and $z = 1.23$.

Evaluate the given expression in Exercises 13–18.

13. $|x| - |x| \cdot |y|$ when $x = -3$, $y = 4$
14. $|x + y| + |x - y|$ when $x = -3$, $y = 2$
15. $\dfrac{|a - 2b|}{2a}$ when $a = 1$, $b = 2$
16. $\dfrac{|x| + |y|}{|x| - |y|}$ when $x = -3$, $y = 4$
17. $\dfrac{-|a - 2b|}{|a + b|}$ when $a = -2$, $b = -1$
18. $\dfrac{|a - b| - 2|c - a|}{|a - b + c|}$ when $a = -2$, $b = 3$, $c = -5$

Carry out the indicated operations in Exercises 19–24.

19. $b^5 \cdot b^2$
20. $x^3 \cdot x^5$
21. $(4y^3)(-5y^6)$
22. $(-6x^4)(-4x^7)$
23. $\left(\dfrac{3}{2}x^3\right)(-2x)$
24. $\left(-\dfrac{5}{3}x^6\right)\left(-\dfrac{3}{10}x^3\right)$
25. Evaluate the given expressions and verify your answer using your calculator.

    a. 1^3 b. 10^8

    c. 2^5 d. 7^1
26. Evaluate the given expressions using your calculator.

    a. 9^{10} b. 0.8^6
27. Which of the following expressions are *not* polynomials?

    a. $-3x^2 + 2x + 5$ b. $-3x^2y$

    c. $-3x^{2/3} + 2xy + 5$ d. $-2x^{-4} + 2xy^3 + 5$

28. Which of the following expressions are *not* polynomials?

a. $4x^5 - x^{1/2} + 6$ b. $\frac{2}{5}x^3 + \frac{4}{3}x - 2$

c. $4x^5y$ d. $x^{4/3}y + 2x - 3$

In Exercises 29–32, indicate the leading coefficient and the degree of the given polynomial.

29. $2x^3 + 3x^2 - 5$ 30. $-4x^5 - 8x^2 + x + 3$

31. $\frac{3}{5}x^4 + 2x^2 - x - 1$

32. $-1.5 + 7x^3 + 0.75x^7$

In Exercises 33–36, find the degree of the given polynomial.

33. $3x^2y - 4x^2 - 2y + 4$

34. $4xy^3 + xy^2 - y^2 + y$

35. $2xy^3 - y^3 + 3x^2 - 2$ 36. $\frac{1}{2}x^3y^3 - 2$

37. Find the value of the polynomial $3x^2y^2 + 2xy - x + 2y + 7$ when $x = 2$ and $y = -1$.

38. Find the value of the polynomial $0.02x^2 + 0.3x - 0.5$ when $x = 0.3$.

39. Find the value of the polynomial $2.1x^3 + 3.3x^2 - 4.1x - 7.2$ when $x = 4.1$.

40. Write a polynomial giving the area of a circle of radius r.

41. Write a polynomial giving the area of a triangle of base b and height h.

42. A field consists of a rectangle and a square arranged as shown in Figure 4. What does each of the following polynomials represent?

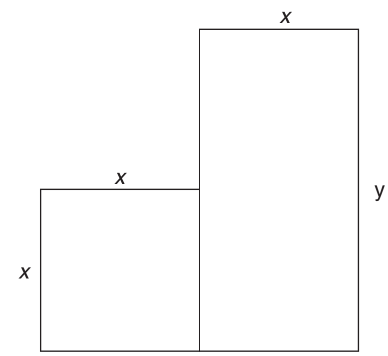


FIGURE 4 See Exercise 42.

a. $x^2 + xy$ b. $2x + 2y$

c. $4x$ d. $4x + 2y$

43. An investor buys x shares of G.E. stock at \$35.5 per share, y shares of Exxon stock at \$91 per share, and z shares of AT&T stock at \$38 per share. What does the polynomial $35.5x + 91y + 38z$ represent?

Perform the indicated operations in Exercises 44–62.

44. $(4x^2 + 3x + 2) + (3x^2 - 2x - 5)$

45. $(2x^2 + 3x + 8) - (5 - 2x + 2x^2)$

46. $4xy^2 + 2xy + 2x + 3 - (-2xy^2 + xy - y + 2)$

47. $(2s^2t^3 - st^2 + st - s + t) - (3s^2t^2 - 2s^2t - 4st^2 - t + 3)$

48. $3xy^2z - 4x^2yz + xy + 3 - (2xy^2z + x^2yz - yz + x - 2)$

49. $a^2bc + ab^2c + 2ab^3 - 3a^2bc - 4ab^3 + 3$

50. $(x + 1)(x^2 + 2x - 3)$

51. $(2 - x)(2x^3 + x - 2)$

52. $(2s - 3)(s^3 - s + 2)$

53. $(-3s + 2)(-2s^2 - s + 3)$

54. $(x^2 + 3)(2x^2 - x + 2)$

55. $(2y^2 + y)(-2y^3 + y - 3)$

56. $(x^2 + 2x - 1)(2x^2 - 3x + 2)$

57. $(a^2 - 4a + 3)(4a^3 + 2a + 5)$

58. $(2a^2 + ab + b^2)(3a - b^2 + 1)$

59. $(-3a + ab + b^2)(3b^2 + 2b + 2)$

60. $5(2x - 3)^2$

61. $2(3x - 2)(3 - x)$

62. $(x - 1)(x + 2)(x + 3)$

63. An investor buys x shares of IBM stock at \$98 per share at Thursday's opening of the stock market. Later in the day, the investor sells y shares of AT&T stock at \$38 per share and z shares of TRW stock at \$20 per share. Write a polynomial that expresses the amount of money the buyer has invested at the end of the day.

64. An artist takes a rectangular piece of cardboard whose sides are x and y and cuts out a square of side $\frac{x}{2}$ to obtain a mat for a painting, as shown in Figure 5. Write a polynomial giving the area of the mat.

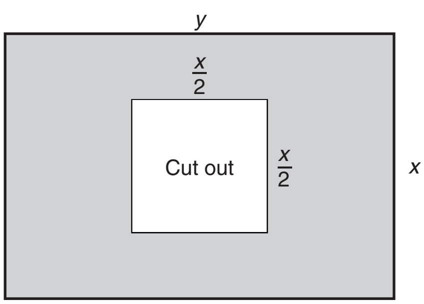


FIGURE 5 See Exercise 64.

In Exercises 65–78, perform the multiplication mentally.

65. $(x - 1)(x + 3)$

66. $(x + 2)(x + 3)$

67. $(2x + 1)(2x + 3)$

68. $(3x - 1)(x + 5)$

69. $(3x - 2)(x - 1)$

70. $(x + 4)(2x - 1)$

71. $(x + y)^2$

72. $(x - 4)^2$

73. $(3x - 1)^2$

74. $(x + 2)(x - 2)$

75. $(2x + 1)(2x - 1)$

76. $(3a + 2b)^2$

77. $(x^2 + y^2)^2$

78. $(x - y)^2$

79. Simplify the following.

a. $3^{10} + 3^{10} + 3^{10}$ b. $2^n + 2^n + 2^n + 2^n$

80. A student conjectured that the expression $N = m^2 - m + 41$ yields N, a prime number, for integer values of m. Prove or disprove this statement.

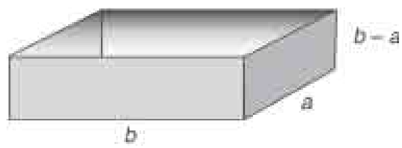
81. Perform the indicated operations.

a. $\left(\frac{2}{x} - 1\right)\left(\frac{2}{x} + 1\right)$

b. $\left(\frac{wx}{y} - z\right)^2$

c. $(x + y + z)(x + y - z)$

82. Find the surface area and volume of the open-top box below.



83. Eric can run a mile in 4.23 minutes, and Benjamin can run 4.23 miles in an hour. Who is the faster runner?

84. a. Let $P = \$1000$; that is, store 1000 in memory location P. Evaluate $P + 0.06P$ by entering the expression $P + 0.06P$ into your calculator.

b. Repeat part (a) for $P = \$28{,}525$.

85. Let $A = 8$ and $B = 32$; that is, store 8 in memory location A and 32 in memory location B. Evaluate the following expressions by entering them into your calculator as they appear below. (Use a multiplication sign if your calculator requires you to do so.)

a. $A(B + 17)$ b. $5B - A^2$

c. A^A d. $16^A - 3AB$

86. Find the value of the polynomial $20t - 0.7t^2$ when t is 28 and when t is 29. Try to find a value for t (other than 0) that gives the expression a value close to zero.

87. Consider the polynomial $vt - \frac{1}{2}at^2$.

a. Compare this expression to the expression given in Exercise 86. What values of v and a would make them identical?

b. Using your calculator, experiment with different values of v, a, and t. Try to put your data in an organized chart. In physics, this expression represents position of a body in free fall: v is the initial velocity, and a is the acceleration due to gravity.

1.4 Factoring

Now that we can find the product of two polynomials, let us consider the reverse problem: given a polynomial, can we find factors whose product yields the given polynomial? This process, known as **factoring**, is one of the basic tools of algebra. In this chapter, a polynomial with *integer* coefficients is to be factored as a product of polynomials of lower degree with *integer* coefficients; a polynomial with *rational* coefficients is to be factored as a product of polynomials of lower degree with *rational* coefficients. We will approach factoring by learning to recognize the situations in which factoring is possible.

Common Factors

Consider the polynomial

$$x^2 + x$$

Since the factor x is common to both terms, we can write

$$x^2 + x = x(x + 1)$$

EXAMPLE 1 FACTORING WITH COMMON FACTORS

Factor.

a. $15x^3 - 10x^2$ b. $4x^2y - 8xy^2 + 6xy$ c. $2x(x + y) - 5y(x + y)$

SOLUTION

a. 5 and x^2 are common to both terms. Therefore,

$$15x^3 - 10x^2 = 5x^2(3x - 2)$$

b. Here we see that 2, x, and y are common to all terms. Therefore,

$$4x^2y - 8xy^2 + 6xy = 2xy(2x - 4y + 3)$$

c. The expression $(x + y)$ is found in both terms. Factoring, we have

$$2x(x + y) - 5y(x + y) = (x + y)(2x - 5y)$$

✔ Progress Check

Factor.

a. $4x^2 - x$ b. $3x^4 - 9x^2$ c. $3m(2x - 3y) - n(2x - 3y)$

Answers

a. $x(4x - 1)$ b. $3x^2(x^2 - 3)$ c. $(2x - 3y)(3m - n)$

Factoring by Grouping

It is sometimes possible to discover common factors by first grouping terms. Consider the following examples:

EXAMPLE 2 FACTORING BY GROUPING

Factor.

a. $2ab + b + 2ac + c$ b. $2x - 4x^2y - 3y + 6xy^2$

SOLUTION

a. Group those terms containing b and those terms containing c.

$2ab + b + 2ac + c$	$= (2ab + b) + (2ac + c)$	Grouping
	$= b(2a + 1) + c(2a + 1)$	Common factors b, c
	$= (2a + 1)(b + c)$	Common factors $2a + 1$

Alternatively, suppose we group terms containing a.

$2ab + b + 2ac + c$	$= (2ab + 2ac) + (b + c)$	Grouping
	$= 2a(b + c) + (b + c)$	Common factors 2, a
	$= (b + c)(2a + 1)$	Common factor $(b + c)$

b. $2x - 4x^2y - 3y + 6xy^2$

$= (2x - 4x^2y) - (3y - 6xy^2)$	Grouping with sign change
$= 2x(1 - 2xy) - 3y(1 - 2xy)$	Common factors $2x$, $3y$
$= (1 - 2xy)(2x - 3y)$	Common factor $1 - 2xy$

✔ Progress Check

Factor.

a. $2m^3n + m^2 + 2mn^2 + n$ b. $2a^2 - 4ab^2 - ab + 2b^3$

Answers

a. $(2mn + 1)(m^2 + n)$ b. $(a - 2b^2)(2a - b)$

Factoring Second-Degree Polynomials

To factor a second-degree polynomial, such as

$$x^2 + 5x + 6$$

we first note that the term x^2 can have come only from $x \cdot x$, so we write two incomplete factors:

$$x^2 + 5x + 6 = (x \qquad)(x \qquad)$$

The constant term +6 can be the product of either two positive numbers or two negative numbers. Since the middle term $+5x$ is the sum of two other products, both signs must be positive. Thus,

$$x^2 + 5x + 6 = (x + \quad)(x + \quad)$$

Finally, the number 6 can be written as the product of two integers in only two ways: $1 \cdot 6$ and $2 \cdot 3$. The first pair gives a middle term of $7x$. The second pair gives the actual middle term, $5x$. So

$$x^2 + 5x + 6 = (x + 2)(x + 3)$$

EXAMPLE 3 FACTORING SECOND-DEGREE POLYNOMIALS

Factor.

a. $x^2 - 7x + 10$ b. $x^2 - 3x - 4$

SOLUTION

a. Since the constant term is positive and the middle term is negative, we must have two negative signs. Integer pairs whose product is 10 are 1 and 10, and 2 and 5. We find that

$$x^2 - 7x + 10 = (x - 2)(x - 5)$$

b. Since the constant term is negative, we must have opposite signs. Integer pairs whose product is 4 are 1 and 4, and 2 and 2. Since the coefficient of $-3x$ is negative, we assign the larger integer of a given pair to be negative. We find that

$$x^2 - 3x - 4 = (x + 1)(x - 4)$$

When the leading coefficient of a second-degree polynomial is an integer other than 1, the factoring process becomes more complex, as shown in the following example.

EXAMPLE 4 FACTORING SECOND-DEGREE POLYNOMIALS

Factor $2x^2 - x - 6$.

SOLUTION

The term $2x^2$ can result only from the factors $2x$ and x, so the factors must be of the form

$$2x^2 - x - 6 = (2x \qquad)(x \qquad)$$

The constant term, -6, must be the product of factors of opposite signs, so we may write

$$2x^2 - x - 6 = \begin{cases} (2x + \quad)(x - \quad) \\ \text{or} \\ (2x - \quad)(x + \quad) \end{cases}$$

The integer factors of 6 are

$$1 \cdot 6 \qquad 6 \cdot 1 \qquad 2 \cdot 3 \qquad 3 \cdot 2$$

By trying these we find that

$$2x^2 - x - 6 = (2x + 3)(x - 2)$$

✔ Progress Check

Factor.

a. $3x^2 - 16x + 21$ b. $2x^2 + 3x - 9$

Answers

a. $(3x - 7)(x - 3)$ b. $(2x - 3)(x + 3)$

WARNING

The polynomial $x^2 - 6x$ can be written as

$$x^2 - 6x = x(x - 6)$$

and is then a product of two polynomials of positive degree. Students often fail to consider x to be a "true" factor.

Special Factors

There is a special case of the second-degree polynomial that occurs frequently and factors easily. Given the polynomial $x^2 - 9$, we see that each term is a perfect square, and we can verify that

$$x^2 - 9 = (x + 3)(x - 3)$$

The general rule, which holds whenever we are dealing with a difference of two squares, may be stated as follows:

Difference of Two Squares

$$a^2 - b^2 = (a + b)(a - b)$$

EXAMPLE 5 SPECIAL FACTORS

Factor.

a. $4x^2 - 25$ b. $9r^2 - 16t^2$

SOLUTION

a. Since

$$4x^2 - 25 = (2x)^2 - (5)^2$$

we may use the formula for the difference of two squares with $a = 2x$ and $b = 5$. Thus,

$$4x^2 - 25 = (2x + 5)(2x - 5)$$

b. Since

$$9r^2 - 16t^2 = (3r)^2 - (4t)^2$$

we have $a = 3r$ and $b = 4t$, resulting in

$$9r^2 - 16t^2 = (3r + 4t)(3r - 4t)$$

✔ Progress Check

Factor.

a. $x^2 - 49$ b. $16x^2 - 9$ c. $25x^2 - y^2$

Answers

a. $(x + 7)(x - 7)$ b. $(4x + 3)(4x - 3)$ c. $(5x + y)(5x - y)$

The formulas for a sum of two cubes and a difference of two cubes can be verified by multiplying the factors on the right-hand sides of the following equations:

Sum and Difference of Two Cubes

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$
$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

These formulas provide a direct means of factoring the sum or difference of two cubes and are used in the same way as the formula for a difference of two squares.

EXAMPLE 6 SPECIAL FACTORS

Factor.

a. $x^3 + 1$ b. $27m^3 - 64n^3$ c. $\frac{1}{27}u^3 + 8v^3$

SOLUTION

a. With $a = x$ and $b = 1$, the formula for the sum of two cubes yields the following result:

$$x^3 + 1 = (x + 1)(x^2 - x + 1)$$

b. Since

$$27m^3 - 64n^3 = (3m)^3 - (4n)^3$$

we can use the formula for the difference of two cubes with $a = 3m$ and $b = 4n$:

$$27m^3 - 64n^3 = (3m - 4n)(9m^2 + 12mn + 16n^2)$$

c. Note that

$$\frac{1}{27}u^3 + 8v^3 = \left(\frac{1}{3}u\right)^3 + (2v)^3$$

and then use the formula for the sum of two cubes:

$$\frac{1}{27}u^3 + 8v^3 = \left(\frac{u}{3} + 2v\right)\left(\frac{u^2}{9} - \frac{2}{3}uv + 4v^2\right)$$

Combining Methods

We conclude with problems that combine the various methods of factoring that we have studied. As the factoring becomes more complicated, it may be helpful to consider the following strategy:

> Remove common factors before
> attempting any other factoring techniques.

EXAMPLE 7 COMMON FACTORS, GROUPING, AND SPECIAL FACTORS

Factor.

a. $2x^3 - 8x$ b. $3y(y + 3) + 2(y + 3)(y^2 - 1)$

SOLUTION

a. Observing the common factor $2x$, we find that

$$\begin{aligned} 2x^3 - 8x &= 2x(x^2 - 4) \\ &= 2x(x+2)(x-2) \end{aligned}$$

b. Observing the common factor $y + 3$, we see that

$$\begin{aligned} 3y(y+3) + 2(y+3)(y^2-1) &= (y+3)[3y + 2(y^2-1)] \\ &= (y+3)(3y + 2y^2 - 2) \\ &= (y+3)(2y^2 + 3y - 2) \\ &= (y+3)(2y-1)(y+2) \end{aligned}$$

Focus on "MAGICAL" Factoring for Second-Degree Polynomials

Factoring involves a certain amount of trial and error that can become frustrating, especially when the lead coefficient is not 1. You might want to try a scheme that "magically" reduces the number of candidates. We demonstrate the method for the polynomial

$$4x^2 + 11x + 6 \tag{1}$$

Using the lead coefficient of 4, write the pair of incomplete factors

$$(4x \qquad)(4x \qquad) \tag{2}$$

Next, multiply the coefficient of x^2 and the constant term in Equation (1) to produce $4 \cdot 6 = 24$. Now find two integers whose product is 24 and whose sum is 11, the coefficient of the middle term of (1). Since 8 and 3 work, we write

$$(4x + 8)(4x + 3) \tag{3}$$

Finally, within each parenthesis in Equation (3) discard any common numerical factor. (Discarding a factor may only be performed in this "magical" type of factoring.) Thus $(4x + 8)$ reduces to $(x + 2)$ and we write

$$(x + 2)(4x + 3) \tag{4}$$

which is the factorization of $4x^2 + 11x + 6$.

Will the method always work? Yes—if you first remove all common factors in the original polynomial. That is, you must first write

$$6x^2 + 15x + 6 = 3(2x^2 + 5x + 2)$$

and apply the method to the polynomial $2x^2 + 5x + 2$.

(For a proof that the method works, see M. A. Autrie and J. D. Austin, "A Novel Way to Factor Quadratic Polynomials," *The Mathematics Teacher* 72, no. 2 [1979].) We use the polynomial $2x^2 - x - 6$ of Example 4 to demonstrate the method when some of the coefficients are negative.

Try the method on these second-degree polynomials:

$$3x^2 + 10x - 8$$
$$6x^2 - 13x + 6$$
$$4x^2 - 15x - 4$$
$$10x^2 + 11x - 6$$

Factoring $ax^2 + bx + c$	Example: $2x^2 - x - 6$
Step 1. Use the lead coefficient a to write the incomplete factors $(ax \quad)(ax \quad)$	*Step 1.* The lead coefficient is 2, so we write $(2x \quad)(2x \quad)$
Step 2. Multiply a and c, the coefficients of x^2, and the constant term.	*Step 2.* $a \cdot c = (2)(-6) = -12$
Step 3. Find integers whose product is $a \cdot c$ and whose sum equals b. Write these integers in the incomplete factors of *Step 1*.	*Step 3.* Two integers whose product is -12 and whose sum is -1 are 3 and -4. We then write $(2x + 3)(2x - 4)$
Step 4. Discard any common factor *within each parenthesis* in *Step 3*. The result is the desired factorization.	*Step 4.* Reducing $(2x - 4)$ to $(x - 2)$ by discarding the common factor 2, we have $2x^2 - x - 6 = (2x + 3)(x - 2)$

✔ Progress Check

Factor.

a. $x^3 + 5x^2 - 6x$ b. $2x^3 - 2x^2y - 4xy^2$

c. $-3x(x + 1) + (x + 1)(2x^2 + 1)$

Answers

a. $x(x + 6)(x - 1)$ b. $2x(x + y)(x - 2y)$ c. $(x + 1)(2x - 1)(x - 1)$

Irreducible Polynomials

Are there polynomials that cannot be written as a product of polynomials of lower degree with integer coefficients? The answer is yes. Examples are the polynomials $x^2 + 1$ and $x^2 + x + 1$. A polynomial is said to be **prime**, or **irreducible**, if it cannot be written as a product of two polynomials, each of positive degree. Thus, $x^2 + 1$ is irreducible over the integers.

Exercise Set 1.4

Factor completely.

1. $5x - 15$
2. $\frac{1}{4}x + \frac{3}{4}y$
3. $-2x - 8y$
4. $3x - 6y + 15$
5. $5bc + 25b$
6. $2x^4 + x^2$
7. $-3y^2 - 4y^5$
8. $3abc + 12bc$
9. $3x^2 + 6x^2y - 9x^2z$
10. $9a^3b^3 + 12a^2b - 15ab^2$
11. $x^2 + 4x + 3$
12. $x^2 + 2x - 8$
13. $y^2 - 8y + 15$
14. $y^2 + 7y - 8$
15. $a^2 - 7ab + 12b^2$
16. $x^2 - 49$
17. $y^2 - \frac{1}{9}$
18. $a^2 - 7a + 10$

19. $9 - x^2$

20. $4b^2 - a^2$

21. $x^2 - 5x - 14$

22. $x^2y^2 - 9$

23. $\frac{1}{16} - y^2$

24. $4a^2 - b^2$

25. $x^2 - 6x + 9$

26. $a^2b^2 - \frac{1}{9}$

27. $x^2 - 12x + 20$

28. $x^2 - 8x - 20$

29. $x^2 + 11x + 24$

30. $y^2 - \frac{9}{16}$

31. $2x^2 - 3x - 2$

32. $2x^2 + 7x + 6$

33. $3a^2 - 11a + 6$

34. $4x^2 - 9x + 2$

35. $6x^2 + 13x + 6$

36. $4y^2 - 9$

37. $8m^2 - 6m - 9$

38. $9x^2 + 24x + 16$

39. $10x^2 - 13x - 3$

40. $9y^2 - 16x^2$

41. $6a^2 - 5ab - 6b^2$

42. $4x^2 + 20x + 25$

43. $10r^2s^2 + 9rst + 2t^2$

44. $x^{12} - 1$

45. $16 - 9x^2y^2$

46. $6 + 5x - 4x^2$

47. $8n^2 - 18n - 5$

48. $15 + 4x - 4x^2$

49. $2x^2 - 2x - 12$

50. $3y^2 + 6y - 45$

51. $30x^2 - 35x + 10$

52. $x^4y^4 - x^2y^2$

53. $18x^2m + 33xm + 9m$

54. $25m^2n^3 - 5m^2n$

55. $12x^2 - 22x^3 - 20x^4$

56. $10r^2 - 5rs - 15s^2$

57. $x^4 - y^4$

58. $a^4 - 16$

59. $b^4 + 2b^2 - 8$

60. $4b^4 + 20b^2 + 25$

61. $x^3 + 27y^3$

62. $8x^3 + 125y^3$

63. $27x^3 - y^3$

64. $64x^3 - 27y^3$

65. $a^3 + 8$

66. $8r^3 - 27$

67. $\frac{1}{8}m^3 - 8n^3$

68. $8a^3 - \frac{1}{64}b^3$

69. $(x + y)^3 - 8$

70. $27 + (x + y)^3$

71. $8x^6 - 125y^6$

72. $a^6 + 27b^6$

73. $4(x + 1)(y + 2) - 8(y + 2)$

74. $2(x + 1)(x - 1) + 5(x - 1)$

75. $3(x + 2)^2(x - 1) - 4(x + 2)^2(2x + 7)$

76. $4(2x - 1)^2(x + 2)^3(x + 1) - 3(2x - 1)^5(x + 2)^2(x + 3)$

77. Show that the difference of the squares of two positive, consecutive odd integers must be divisible by 8.

78. A perfect square is a natural number of the form n^2. For example, 9 is a perfect square since $9 = 3^2$. Show that the sum of the squares of two odd numbers cannot be a perfect square.

79. Find a natural number n, if possible, such that $1 + n(n + 1)(n + 2)(n + 3)$ is a perfect square.

80. Prove or disprove that $1 + n(n + 1)(n + 2)(n + 3)$ is a perfect square. (*Hint:* Consider $[1 + n(n + 3)]^2$.)

81. Factor completely.

a. $(x + h)^3 - x^3$

b. $2^n + 2^{n+1} + 2^{n+2}$

c. $16 - 81x^{12}$

d. $z^2 - x^2 + 2xy - y^2$

82. Factor completely.

a. $\left[\frac{n(n + 1)}{2}\right]^2 + (n + 1)^3$

b. $\frac{n(n + 1)(2n + 1)}{6} + (n + 1)^2$

c. $\frac{1}{b}(a + bx)^2 - \frac{a}{b}(a + bx)$

83. Factor the following expressions that arise in different branches of science.

a. biology (blood flow): $C[(R + 1)^2 - r^2]$

b. physics (nuclear): $pa^2 + (1 - p)b^2 - [pa + (1 - p)b]^2$

c. mechanics (bending beams): $X^2 - 3LX + 2L^2$

d. electricity (resistance): $(R_1 + R_2)^2 - 2r(R_1 + R_2)$

e. physics (motion): $-16t^2 + 64t + 336$

84. a. Factor this expression, used to find the answers given in the chapter opening.

$$20t - 0.7t^2$$

b. Show that the factored form and the original form are identical by using your graphing calculator to compare the GRAPH of each expression. Graphing will be explained in Chapter 3.

85. Factor the general expression $vt - \frac{1}{2}at^2$.

86. Suppose we alter the expression from Exercise 85 by adding a constant:

$$s + vt - \frac{1}{2}at^2$$

Experiment with different values of s, v, and t. Which ones give you an expression that is easy to factor? (Reread Exercise 87 in Section 1.3. The s we have added could represent the original position of the object.)

87. *Mathematics in Writing:* Write a short paragraph explaining the differences in the techniques you used to factor the scientific expressions in Exercise 83 parts a, b, and c. Find at least one other problem in this problem set that uses a technique similar to each of the three you have described.

1.5 Rational Expressions

Much of the terminology and many of the techniques for the arithmetic of fractions carry over to **algebraic fractions**, which are the quotients of algebraic expressions. In particular, we refer to a quotient of two polynomials as a **rational expression**.

Notation

Any symbol used as a divisor in this text is always assumed to be different from zero.

Therefore, we will not always identify a divisor as being different from zero unless it disappears through some type of mathematical manipulation.

Our objective in this section is to review the procedures for adding, subtracting, multiplying, and dividing rational expressions. We are then able to convert a complicated fraction, such as

$$\frac{1 - \frac{1}{x}}{\frac{1}{x^2} + \frac{1}{x}}$$

into a form that simplifies evaluation of the fraction and facilitates other operations with it.

Multiplication and Division of Rational Expressions

The symbols appearing in rational expressions represent real numbers. We may, therefore, apply the rules of arithmetic to rational expressions. Let a, b, c, and d represent any algebraic expressions.

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$ Multiplication of rational expressions

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{\frac{a}{b}}{\frac{c}{d}} \cdot \frac{(bd)}{(bd)} = \frac{ad}{cb}$$ Division of rational expressions

EXAMPLE 1 MULTIPLICATION AND DIVISION OF RATIONAL EXPRESSIONS

Divide $\frac{2x}{y}$ by $\frac{3y^3}{x-3}$.

SOLUTION

$$\frac{\frac{2x}{y}}{\frac{3y^3}{x-3}} = \frac{\frac{2x}{y}}{\frac{3y^3}{x-3}} \cdot \frac{(y)(x-3)}{(y)(x-3)} = \frac{2x(x-3)}{3y^4}, \quad x \neq 3$$

The basic rule that allows us to simplify rational expressions is the cancellation principle.

Cancellation Principle

$$\frac{ab}{ac} = \frac{b}{c}, \quad a \neq 0$$

This rule results from the fact that $\frac{a}{a} = 1$. Thus,

$$\frac{ab}{ac} = \frac{a}{a} \cdot \frac{b}{c} = 1 \cdot \frac{b}{c} = \frac{b}{c}$$

Once again we find that a rule for the arithmetic of fractions carries over to rational expressions.

EXAMPLE 2 FACTORING AND CANCELLATION

Simplify.

a. $\dfrac{x^2-4}{x^2+5x+6}$ b. $\dfrac{3x^2(y-1)}{y+1} \div \dfrac{6x(y-1)^2}{(y+1)^3}$ c. $\dfrac{x^2-x-6}{3x-x^2}$

SOLUTION

a. $$\frac{x^2-4}{x^2+5x+6} = \frac{(x+2)(x-2)}{(x+3)(x+2)} = \frac{x-2}{x+3}, \qquad x \neq -2$$

b. $$\frac{\dfrac{3x^2(y-1)}{y+1}}{\dfrac{6x(y-1)^2}{(y+1)^3}} = \frac{\dfrac{3x^2(y-1)}{y+1}}{\dfrac{6x(y-1)^2}{(y+1)^3}} \cdot \frac{(y+1)^3}{(y+1)^3} = \frac{3x^2(y-1)(y+1)^2}{6x(y-1)^2}$$

$$= \frac{x(y+1)^2}{2(y-1)}, \qquad x \neq 0, \ y \neq -1$$

c. $$\frac{x^2-x-6}{3x-x^2} = \frac{(x-3)(x+2)}{x(3-x)} = \frac{(x-3)(x+2)}{-x(x-3)} = \frac{x+2}{-x}, \qquad x \neq 3$$

$$= -\frac{x+2}{x}, \qquad x \neq 3$$

Note that in Example 2c, we wrote $(3-x)$ as $-(x-3)$. This technique is often used to recognize factors that may be canceled.

✔ Progress Check

Simplify.

a. $\dfrac{4-x^2}{x^2-x-6}$ b. $\dfrac{8-2x}{y} \div \dfrac{x^2-16}{y}$

Answers

a. $\dfrac{2-x}{x-3}$, $x \neq -2$ b. $-\dfrac{2}{x+4}$, $x \neq 4$, $y \neq 0$

WARNING

a. Only multiplicative factors can be canceled. Thus,

$$\frac{2x-4}{x} \neq 2-4$$

Since x is *not a factor* in the numerator, we may *not* perform cancellation.

b. Note that

$$\frac{y^2 - x^2}{y - x} \neq y - x$$

To simplify correctly, write

$$\frac{y^2 - x^2}{y - x} = \frac{(y + x)(y - x)}{y - x} = y + x, \qquad y \neq x$$

Addition and Subtraction of Rational Expressions

Since the variables in rational expressions represent real numbers, the rules of arithmetic for addition and subtraction of fractions apply to rational expressions. When rational expressions have the same denominator, the addition and subtraction rules are as follows:

$$\frac{a}{c} + \frac{b}{c} = \frac{a + b}{c}$$

$$\frac{a}{c} - \frac{b}{c} = \frac{a - b}{c}$$

For example,

$$\frac{2}{x - 1} - \frac{4}{x - 1} + \frac{5}{x - 1} = \frac{2 - 4 + 5}{x - 1} = \frac{3}{x - 1}$$

To add or subtract rational expressions with *different* denominators, we must first rewrite each rational expression as an equivalent one with the same denominator as the others. Although any common denominator will do, we will concentrate on finding the **least common denominator**, or **LCD**, of two or more rational expressions. We now outline the procedure and provide some examples.

EXAMPLE 3 LEAST COMMON DENOMINATOR FOR RATIONAL NUMBERS

Find the LCD of the following three fractions:

$$\frac{1}{8} \qquad \frac{7}{90} \qquad \frac{3}{25}$$

SOLUTION

Least Common Denominator

Method	Example
Step 1. Factor the denominator of each fraction.	*Step 1.* $\frac{1}{2^3}$ $\quad \frac{7}{2 \cdot 3^2 \cdot 5}$ $\quad \frac{3}{5^2}$
Step 2. Find the different factors in the denominator and the highest power to which each factor occurs.	*Step 2.*
Step 3. The product of the final factors in *Step 2* is the LCD.	*Step 3.* The LCD is $2^3 \cdot 3^2 \cdot 5^2 = 8 \cdot 9 \cdot 25$.

Factor	Highest exponent	Final factors
2	3	8
3	2	9
5	2	25

EXAMPLE 4 **LEAST COMMON DENOMINATOR FOR RATIONAL EXPRESSIONS**

Find the LCD of the following three rational expressions:

$$\frac{1}{x^3 - x^2} \qquad \frac{-2}{x^3 - x} \qquad \frac{3x}{x^2 + 2x + 1}$$

SOLUTION

Least Common Denominator

Method	Example
Step 1. Factor the denominator of each fraction.	*Step 1.* $\frac{1}{x^2(x-1)}$ $\quad \frac{-2}{x(x-1)(x+1)}$ $\quad \frac{3x}{(x+1)^2}$
Step 2. Find the different factors in the denominator and the highest power to which each factor occurs.	*Step 2.*
Step 3. The product of the final factors in *Step 2* is the LCD.	*Step 3.* The LCD is $x^2(x-1)(x+1)^2$

Factor	Highest exponent	Final factors
x	2	x^2
$x - 1$	1	$x - 1$
$x + 1$	2	$(x + 1)^2$

✔ Progress Check

Find the LCD of the following fractions:

$$\frac{2a}{(3a^2 + 12a + 12)b} \qquad \frac{-7b}{a(4b^2 - 8b + 4)} \qquad \frac{3}{ab^3 + 2b^3}$$

Answers

$12ab^3(a + 2)^2(b - 1)^2$

Equivalent Fractions

The fractions $\frac{2}{5}$ and $\frac{6}{15}$ are said to be **equivalent** because we obtain $\frac{6}{15}$ by multiplying $\frac{2}{5}$ by $\frac{3}{3}$, which is the same as multiplying $\frac{2}{5}$ by 1. We also say that algebraic fractions are **equivalent fractions** if we can obtain one from the other by multiplying both the numerator and denominator by the same expression.

To add rational expressions, we must first determine the LCD and then convert each rational expression into an equivalent fraction with the LCD as its denominator. We can accomplish this conversion by multiplying the fraction by the appropriate equivalent of 1. We now outline the procedure and provide an example.

EXAMPLE 5 ADDITION AND SUBTRACTION OF RATIONAL EXPRESSIONS

Simplify.

$$\frac{x+1}{2x^2} - \frac{2}{3x(x+2)}$$

SOLUTION

Addition of Rational Expressions

Method	Example
Step 1. Find the LCD.	*Step 1.* $\text{LCD} = 6x^2(x+2)$
Step 2. Multiply each rational expression by a fraction whose numerator and denominator are the same, and consist of all factors of the LCD that are missing in the denominator of the expression.	*Step 2.* $\frac{x+1}{2x^2} \cdot \frac{3(x+2)}{3(x+2)} = \frac{3x^2+9x+6}{6x^2(x+2)}$ $\frac{2}{3x(x+2)} \cdot \frac{(2x)}{(2x)} = \frac{4x}{6x^2(x+2)}$
Step 3. Add the rational expressions. Do not multiply out the denominators since it may be possible to cancel.	*Step 3.* $\frac{x+1}{2x^2} - \frac{2}{3x(x+2)}$ $= \frac{3x^2+9x+6}{6x^2(x+2)} - \frac{4x}{6x^2(x+2)}$ $= \frac{3x^2+5x+6}{6x^2(x+2)}$

✔ Progress Check

Perform the indicated operations.

a. $\frac{x-8}{x^2-4} + \frac{3}{x^2-2x}$ b. $\frac{4r-3}{9r^3} - \frac{2r+1}{4r^2} + \frac{2}{3r}$

Answers

a. $\frac{x-3}{x(x+2)}$, $x \neq 2$ b. $\frac{6r^2+7r-12}{36r^3}$

Complex Fractions

At the beginning of this section, we said that we wanted to be able to simplify fractions such as

$$\frac{1-\dfrac{1}{x}}{\dfrac{1}{x^2}+\dfrac{1}{x}}$$

This is an example of a **complex fraction**, which is a fractional form with fractions in the numerator or denominator or both.

EXAMPLE 6 DIVISION OF RATIONAL EXPRESSIONS

Simplify the following:

$$\frac{1-\dfrac{1}{x}}{\dfrac{1}{x^2}+\dfrac{1}{x}}$$

SOLUTION

Simplifying Complex Fractions

Method	Example
Step 1. Find the LCD of all fractions in the numerator and denominator.	*Step 1.* The LCD of $\frac{1}{x}$ and $\frac{1}{x^2}$ is x^2.
Step 2. Multiply the numerator and denominator by the LCD. Since this is multiplication by 1, the result is an equivalent fraction.	*Step 2.* $\dfrac{\left(1-\frac{1}{x}\right)}{\left(\frac{1}{x^2}+\frac{1}{x}\right)}\cdot\dfrac{(x^2)}{(x^2)}=\dfrac{x^2-x}{1+x},\quad x\neq 0$ $=\dfrac{x(x-1)}{1+x},\quad x\neq 0$

✔ Progress Check

Simplify.

a. $$\frac{2+\dfrac{1}{x}}{1-\dfrac{2}{x}}$$

b. $$\frac{\dfrac{a}{b}+\dfrac{b}{a}}{\dfrac{1}{a}-\dfrac{1}{b}}$$

Answers

a. $\dfrac{2x+1}{x-2},\quad x\neq 0$

b. $\dfrac{a^2+b^2}{b-a},\quad a\neq 0,\quad b\neq 0$

Exercise Set 1.5

Perform all possible simplifications in Exercises 1–20.

1. $\dfrac{x+4}{x^2-16}$
2. $\dfrac{y^2-25}{y+5}$
3. $\dfrac{x^2-8x+16}{x-4}$
4. $\dfrac{5x^2-45}{2x-6}$
5. $\dfrac{6x^2-x-1}{2x^2+3x-2}$
6. $\dfrac{2x^3+x^2-3x}{3x^2-5x+2}$
7. $\dfrac{2}{3x-6} \div \dfrac{3}{2x-4}$
8. $\dfrac{5x+15}{8} \div \dfrac{3x+9}{4}$
9. $\dfrac{25-a^2}{b+3} \cdot \dfrac{2b^2+6b}{a-5}$
10. $\dfrac{2xy^2}{x+y} \cdot \dfrac{x+y}{4xy}$
11. $\dfrac{x+2}{3y} \div \dfrac{x^2-2x-8}{15y^2}$
12. $\dfrac{3x}{x+2} \div \dfrac{6x^2}{x^2-x-6}$
13. $\dfrac{6x^2-x-2}{2x^2-5x+3} \cdot \dfrac{2x^2-7x+6}{3x^2+x-2}$
14. $\dfrac{6x^2+11x-2}{4x^2-3x-1} \cdot \dfrac{5x^2-3x-2}{3x^2+7x+2}$
15. $(x^2-4) \cdot \dfrac{2x+3}{x^2+2x-8}$
16. $(a^2-2a) \cdot \dfrac{a+1}{6-a-a^2}$
17. $(x^2-2x-15) \div \dfrac{x^2-7x+10}{x^2+1}$
18. $\dfrac{2y^2-5y-3}{y-4} \div (y^2+y-12)$
19. $\dfrac{x^2-4}{x^2+2x-3} \cdot \dfrac{x^2+3x-4}{x^2-7x+10} \cdot \dfrac{x+3}{x^2+3x+2}$
20. $\dfrac{x^2-9}{6x^2+x-1} \cdot \dfrac{2x^2+5x+2}{x^2+4x+3} \cdot \dfrac{x^2-x-2}{x^2-3x}$

In Exercises 21–30, find the LCD.

21. $\dfrac{4}{x}, \dfrac{x-2}{y}$
22. $\dfrac{x}{x-1}, \dfrac{x+4}{x+2}$
23. $\dfrac{5-a}{a}, \dfrac{7}{2a}$
24. $\dfrac{x+2}{x}, \dfrac{x-2}{x^2}$
25. $\dfrac{2b}{b-1}, \dfrac{3}{(b-1)^2}$
26. $\dfrac{2+x}{x^2-4}, \dfrac{3}{x-2}$
27. $\dfrac{4x}{x-2}, \dfrac{5}{x^2+x-6}$
28. $\dfrac{3}{y^2-3y-4}, \dfrac{2y}{y+1}$
29. $\dfrac{3}{x+1}, \dfrac{2}{x}, \dfrac{x}{x-1}$
30. $\dfrac{4}{x}, \dfrac{3}{x-1}, \dfrac{x}{x^2-2x+1}$

In Exercises 31–50, perform the indicated operations and simplify.

31. $\dfrac{8}{a-2} + \dfrac{4}{2-a}$
32. $\dfrac{x}{x^2-4} + \dfrac{2}{4-x^2}$
33. $\dfrac{x-1}{3} + 2$
34. $\dfrac{1}{x-1} + \dfrac{2}{x-2}$
35. $\dfrac{1}{a+2} + \dfrac{3}{a-2}$
36. $\dfrac{a}{8b} - \dfrac{b}{12a}$
37. $\dfrac{4}{3x} - \dfrac{5}{xy}$
38. $\dfrac{4x-1}{6x^3} + \dfrac{2}{3x^2}$
39. $\dfrac{5}{2x+6} - \dfrac{x}{x+3}$
40. $\dfrac{x}{x-y} - \dfrac{y}{x+y}$
41. $\dfrac{5x}{2x^2-18} + \dfrac{4}{3x-9}$
42. $\dfrac{4}{r} - \dfrac{3}{r+2}$
43. $\dfrac{1}{x-1} + \dfrac{2x-1}{(x-2)(x+1)}$
44. $\dfrac{2x}{2x+1} - \dfrac{x-1}{(2x+1)(x-2)}$
45. $\dfrac{2x}{x^2+x-2} + \dfrac{3}{x+2}$
46. $\dfrac{2}{x-2} + \dfrac{x}{x^2-x-6}$
47. $\dfrac{2x-1}{x^2+5x+6} - \dfrac{x-2}{x^2+4x+3}$
48. $\dfrac{2x-1}{x^3-4x} - \dfrac{x}{x^2+x-2}$

49. $\dfrac{2x}{x^2-1}+\dfrac{x+1}{x^2+3x-4}$

50. $\dfrac{2x}{x+2}+\dfrac{x}{x-2}-\dfrac{1}{x^2-4}$

In Exercises 51–66, simplify the complex fraction and perform all indicated operations.

51. $\dfrac{1+\dfrac{2}{x}}{1-\dfrac{3}{x}}$

52. $\dfrac{x-\dfrac{1}{x}}{2+\dfrac{1}{x}}$

53. $\dfrac{x+1}{1-\dfrac{1}{x}}$

54. $\dfrac{1-\dfrac{r^2}{s^2}}{1+\dfrac{r}{s}}$

55. $\dfrac{x^2-16}{\dfrac{1}{4}-\dfrac{1}{x}}$

56. $\dfrac{\dfrac{a}{a-b}-\dfrac{b}{a+b}}{a^2-b^2}$

57. $2-\dfrac{1}{1+\dfrac{1}{a}}$

58. $\dfrac{\dfrac{4}{x^2-4}+1}{\dfrac{x}{x^2+x-6}}$

59. $\dfrac{\dfrac{a}{b}-\dfrac{b}{a}}{\dfrac{1}{a}+\dfrac{1}{b}}$

60. $\dfrac{\dfrac{x}{x-2}-\dfrac{x}{x+2}}{\dfrac{2x}{x-2}+\dfrac{x^2}{x-2}}$

61. $3-\dfrac{2}{1-\dfrac{1}{1+x}}$

62. $2+\dfrac{3}{1+\dfrac{2}{1-x}}$

63. $\dfrac{y-\dfrac{1}{1-\dfrac{1}{y}}}{y+\dfrac{1}{1+\dfrac{1}{y}}}$

64. $1-\dfrac{1-\dfrac{1}{y}}{y-\dfrac{1}{y}}$

65. $1-\dfrac{1}{1+\dfrac{1}{1-\dfrac{1}{1+x}}}$

66. $1+\dfrac{1}{1-\dfrac{1}{1+\dfrac{1}{1+x}}}$

67. Combine and simplify.

a. $\dfrac{1}{R_1}+\dfrac{1}{R_2}+\dfrac{1}{R_3}+\dfrac{1}{R_4}$

b. $\dfrac{3}{c-d}+\dfrac{4d}{(c-d)^2}-\dfrac{5d^2}{(c-d)^3}$

c. $\dfrac{6}{k+3}+k-2$

68. Simplify the complex fractions.

a. $\dfrac{\dfrac{1}{(x+h)^2}-\dfrac{1}{x^2}}{h}$

b. $\dfrac{\dfrac{1}{x^2}-\dfrac{1}{9}}{x-3}$

69. Find the errors in the following and correct the statements.

a. $\dfrac{\dfrac{1}{b}}{\dfrac{1}{a}+\dfrac{1}{b}}=\dfrac{1}{\dfrac{1}{a}}$

b. $\dfrac{\dfrac{1}{b}}{\dfrac{1}{a}+\dfrac{1}{b}}=\left(\dfrac{1}{b}\right)\left(\dfrac{a}{1}+\dfrac{b}{1}\right)$

c. $\dfrac{a^2(3b+4a)}{a^2+b^2}=\dfrac{3b+4a}{b^2}$

d. $\dfrac{1-x}{1+x}=-1$

e. $(x^2-y^2)^2=x^4-y^4$

f. $\dfrac{a+b}{b}=a$

1.6 Integer Exponents

Positive Integer Exponents

In Section 1.3, we defined a^n for a real number a and a positive integer n as

$$a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ factors}}$$

and we showed that if m and n are positive integers then $a^m a^n = a^{m+n}$. The method we used to establish this rule was to write out the factors of a^m and a^n and count the total number of occurrences of a. The same method can be used to establish the rest of the properties in Table 9 when m and n are positive integers.

TABLE 9 Properties of Positive Integer Exponents, $m > 0$ and $n > 0$

Example	Property
$2^2 \cdot 2^3 = 2^5$ $4 \cdot 8 = 32$	$a^m a^n = a^{m+n}$
$(2^3)^2 = 2^{3 \cdot 2} = 2^6$ $8^2 = 64$	$(a^m)^n = a^{mn}$
$(2 \cdot 3)^2 = 2^2 \cdot 3^2$ $6^2 = 4 \cdot 9$	$(ab)^m = a^m b^m$
$\left(\frac{6}{2}\right)^2 = \frac{6^2}{2^2}$ $3^2 = \frac{36}{4}$	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$
$\frac{2^5}{2^2} = 2^3$ $\frac{32}{4} = 8$	$\frac{a^m}{a^n} = a^{m-n}$ if $m > n$, $a \neq 0$
$\frac{2^2}{2^5} = \frac{1}{2^3}$ $\frac{4}{32} = \frac{1}{8}$	$\frac{a^n}{a^m} = \frac{1}{a^{m-n}}$ if $m > n$, $a \neq 0$
$\frac{5^2}{5^2} = \frac{25}{25} = 1$	$\frac{a^m}{a^m} = 1$ if $a \neq 0$

EXAMPLE 1 MULTIPLICATION WITH POSITIVE INTEGER EXPONENTS

Simplify the following.

a. $(4a^2b^3)(2a^3b)$ b. $(2x^2y)^4$

SOLUTION

a. $(4a^2b^3)(2a^3b) = 4 \cdot 2 \cdot a^2a^3b^3b = 8a^5b^4$

b. $(2x^2y)^4 = 2^4(x^2)^4y^4 = 16x^8y^4$

✔ Progress Check

Simplify, using only positive exponents.

a. $(x^3)^4$ b. $x^4(x^2)^3$ c. $\dfrac{a^{14}}{a^8}$ d. $\dfrac{-2(x+1)^n}{(x+1)^{2n}}$

e. $(3ab^2)^3$ f. $\left(\dfrac{-ab^2}{c^3}\right)^3$

Answers

a. x^{12} b. x^{10} c. a^6 d. $-\dfrac{2}{(x+1)^n}$

e. $27a^3b^6$ f. $-\dfrac{a^3b^6}{c^9}$

Zero and Negative Exponents

We next expand our rules to include zero and negative exponents when the base is nonzero. We wish to define a^0 to be consistent with the previous rules for exponents. For example, applying the rule $a^ma^n = a^{m+n}$ yields

$$a^ma^0 = a^{m+0} = a^m$$

Dividing both sides by a^m, we obtain $a^0 = 1$. We therefore *define* a^0 for any nonzero real number by

$$a^0 = 1, \quad a \neq 0$$

The same approach leads us to a definition of negative exponents. For consistency, if $m > 0$, we must have

$$a^ma^{-m} = a^{m-m} = a^0 = 1 \qquad \text{or} \qquad a^ma^{-m} = 1 \tag{1}$$

Division of both sides of Equation (1) by a^m suggests that we define a^{-m} as

$$a^{-m} = \frac{1}{a^m}, \quad a \neq 0$$

Dividing Equation (1) by a^{-m}, we have

$$a^m = \frac{1}{a^{-m}}, \quad a \neq 0$$

Thus, a^m and a^{-m} are reciprocals of one another. The rule for handling negative exponents can be expressed as follows:

A nonzero factor moves from numerator to denominator (or from denominator to numerator) by changing the sign of the exponent.

WARNING

It is important to note that

0^0 is not defined. Furthermore, 0^{-m} is also not defined for $m > 0$.

We may also conclude that

$$a^0 \neq 0 \quad \text{and} \quad a^{-m} \neq 0, \quad m > 0$$

We summarize these results in Table 10.

TABLE 10 Properties of Integer Exponents, $a \neq 0$

Example	Property
$\left(-\frac{1}{2}\right)^0 = 1$	$a^0 = 1$
$8 = 2^3 = \frac{1}{2^{-3}}$	$a^m = \frac{1}{a^{-m}}$
$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$	$a^{-m} = \frac{1}{a^m}$

EXAMPLE 2 OPERATIONS WITH INTEGER EXPONENTS

Simplify the following, using only positive exponents:

a. $\dfrac{2}{(x-1)^0}$ b. $(x^2y^{-3})^{-5}$ c. $\dfrac{y^{3k+1}}{2y^{2k}}$, $k > 0$

SOLUTION

a. $\dfrac{2}{(x-1)^0} = \dfrac{2}{1} = 2$ b. $(x^2y^{-3})^{-5} = (x^2)^{-5}(y^{-3})^{-5} = x^{-10}y^{15} = \dfrac{y^{15}}{x^{10}}$

c. $\dfrac{y^{3k+1}}{2y^{2k}} = \dfrac{y^{3k+1}y^{-2k}}{2} = \dfrac{y^{3k+1-2k}}{2} = \dfrac{1}{2}y^{k+1}$

✔ Progress Check

Simplify, using only positive exponents.

a. $x^{-2}y^{-3}$ b. $\dfrac{-3x^4y^{-2}}{9x^{-8}y^6}$ c. $\left(\dfrac{x^{-3}}{x^{-4}}\right)^{-1}$

Answers

a. $\dfrac{1}{x^2y^3}$ b. $-\dfrac{x^{12}}{3y^8}$ c. $\dfrac{1}{x}$

WARNING

Do not confuse negative numbers and negative exponents.

a. $2^{-4} = \dfrac{1}{2^4}$

Note that $2^{-4} \neq -2^4$.

b. $(-2)^{-3} = \dfrac{1}{(-2)^3} = \dfrac{1}{-8} = -\dfrac{1}{8}$

Note that $(-2)^{-3} \neq \dfrac{1}{2^3} = \dfrac{1}{8}$.

Scientific Notation

One of the significant applications of integer exponents is that of scientific notation. This technique enables us to recognize the size of extremely large and extremely small numbers rather quickly and in a more concise form.

Consider the following examples for powers of 10:

$$\text{One thousand} = 1000 = 1 \times 10^3 = 10^3$$

$$\text{One thousandth} = 0.001 = \frac{1}{1000} = \frac{1}{10^3} = 10^{-3}$$

Reversing the procedure we obtain:

1. $10^2 = 1 \times 10^2 = 100.0$, namely, 1 with the decimal point *two* places to the *right* of it.
2. $10^{-2} = 1 \times 10^{-2} = \dfrac{1}{100} = 0.01$, namely, 1 with the decimal point *two* places to the *left* of it.

Scientific Notation

A number is written in scientific notation if it is of the form $\pm a \times 10^m$, where $1 \le a < 10$ and m is some integer. If $a = 1$, it is generally omitted.

EXAMPLE 3 WRITING IN SCIENTIFIC NOTATION

An angstrom (Å) equals 1 ten-billionth of a meter. Write this in scientific notation.

SOLUTION

1 Å = 0.0000000001 meters = 10^{-10} meters

EXAMPLE 4 WRITING IN SCIENTIFIC NOTATION

One light-year is approximately 6 trillion miles. Write this in scientific notation.

SOLUTION

1 light-year ≈ 6,000,000,000,000 miles = 6×10^{12} miles

If $\pm a \times 10^m$ is the result of some calculations or measurements involving scientific notation, then the number of digits present in a are generally taken as the significant digits of the answer. For example, 6×10^{12} has one significant digit.

If we write a number in scientific notation with fewer significant digits than the original number presented, we must round the last significant digit used according to the following rule:

- Add 1 to the last significant digit if the digit following it in the original number is 5, 6, 7, 8, or 9.
- Leave the last significant digit alone if the digit following it in the original number is 0, 1, 2, 3, or 4.

EXAMPLE 5 WRITING IN SCIENTIFIC NOTATION

The speed of light is approximately 186,282 miles per second. Write it in scientific notation with four significant digits.

SOLUTION

The speed of light ≈ 1.863×10^5 miles per second since 186,282 is rounded to 186,300.

EXAMPLE 6 WRITING IN SCIENTIFIC NOTATION

There are 31,557,600 seconds in an average year (365.25 days). Write it in scientific notation with four significant digits.

SOLUTION

1 year = 3.156×10^7 seconds.

EXAMPLE 7 CALCULATIONS IN SCIENTIFIC NOTATION

Find the number of miles in 1 light-year to four significant digits.

SOLUTION

One light-year is the number of miles light travels in 1 year.

$$\begin{aligned} \text{1 light-year} &= (1.863 \times 10^5 \text{ miles per second})(3.156 \times 10^7 \text{ seconds}) \\ &= 5.880 \times 10^{12} \text{ miles.} \end{aligned}$$

Note that this becomes 6×10^{12} miles if we require only one significant digit. The number of significant digits of our answer equals the minimum number of significant digits involved in our calculations.

Calculator Alert

Your calculator has a key that can be used to enter numbers in scientific notation. This key may be labeled $\boxed{\text{EXP}}$, $\boxed{\text{EE}}$, or $\boxed{\text{EEX}}$. Note how numbers entered in this manner appear on the calculator's display window.

Example: 1.863 $\boxed{\text{EXP}}$ 5 × 3.156 $\boxed{\text{EXP}}$ 7 = $\boxed{\text{5.879628 12}}$ or $\boxed{\text{5.879628E 12}}$
≈ 5.880×10^{12}

✔ Progress Check

1 ounce = 0.02834952 kilogram

Write this number in scientific notation using:

a. two significant digits

b. three significant digits

c. five significant digits

Answers

a. 2.8×10^{-2} b. 2.83×10^{-2} c. 2.8350×10^{-2}

Exercise Set 1.6

In Exercises 1–6, the right-hand side is incorrect. Find the correct term.

1. $x^2 \cdot x^4 = x^8$
2. $(y^2)^5 = y^7$
3. $\dfrac{b^6}{b^2} = b^3$
4. $\dfrac{x^2}{x^6} = x^4$
5. $(2x)^4 = 2x^4$
6. $\left(\dfrac{4}{3}\right)^4 = \dfrac{4}{3^4}$

In Exercises 7–64, use the rules for exponents to simplify. Write the answers using only positive exponents.

7. $\left(-\dfrac{1}{2}\right)^4\left(-\dfrac{1}{2}\right)^3$
8. $(x^m)^{3m}$
9. $(y^4)^{2n}$
10. $\dfrac{(-4)^6}{(-4)^{10}}$
11. $-\left(\dfrac{x}{y}\right)^3$
12. $-3r^2r^3$
13. $(x^3)^5 \cdot x^4$
14. $\dfrac{x^{12}}{x^8}$
15. $(-2x^2)^5$
16. $-(2x^2)^5$
17. $x^{3n} \cdot x^n$
18. $(-2)^m(-2)^n$
19. $\dfrac{x^n}{x^{n+2}}$
20. $\left(\dfrac{3x^3}{y^2}\right)^5$
21. $(-5x^3)(-6x^5)$
22. $(x^2)^3(y^2)^4(x^3)^7$
23. $\dfrac{(r^2)^4}{(r^4)^2}$
24. $[(3b+1)^5]^5$
25. $\left(\dfrac{3}{2}x^2y^3\right)^n$
26. $\dfrac{(-2a^2b)^4}{(-3ab^2)^3}$
27. $(2x+1)^3(2x+1)^7$
28. $\dfrac{y^3(y^3)^4}{(y^4)^6}$
29. $(-2a^2b^3)^{2n}$
30. $\left(-\dfrac{2}{3}a^2b^3c^2\right)^3$
31. $2^0 + 3^{-1}$
32. $(xy)^0 - 2^{-1}$
33. $\dfrac{3}{(2x^2+1)^0}$
34. $(-3)^{-3}$
35. $\dfrac{1}{3^{-4}}$
36. x^{-5}
37. $(-x)^3$
38. $-x^{-5}$
39. $\dfrac{1}{y^{-6}}$
40. $(2a)^{-6}$
41. $5^{-3}5^5$
42. $4y^5y^{-2}$
43. $(3^2)^{-3}$
44. $(x^{-2})^4$
45. $(x^{-3})^{-3}$
46. $[(x+y)^{-2}]^2$
47. $\dfrac{2^2}{2^{-3}}$
48. $\dfrac{x^8}{x^{-10}}$
49. $\dfrac{2x^4y^{-2}}{x^2y^{-3}}$
50. $(x^4y^{-2})^{-1}$
51. $(3a^{-2}b^{-3})^{-2}$
52. $\dfrac{1}{(2xy)^{-2}}$
53. $\left(-\dfrac{1}{2}x^3y^{-4}\right)^{-3}$
54. $\dfrac{(x^{-2})^2}{(3y^{-2})^3}$
55. $\dfrac{3a^5b^{-2}}{9a^{-4}b^2}$
56. $\left(\dfrac{x^3}{x^{-2}}\right)^2$

57. $\left(\dfrac{2a^2b^{-4}}{a^{-3}c^{-3}}\right)^2$

58. $\dfrac{2x^{-3}y^2}{x^{-3}y^{-3}}$

59. $(a - 2b^2)^{-1}$

60. $\left(\dfrac{y^{-2}}{y^{-3}}\right)^{-1}$

61. $\dfrac{(a + b)^{-1}}{(a - b)^{-2}}$

62. $(a^{-1} + b^{-1})^{-1}$

63. $\dfrac{a^{-1} + b^{-1}}{a^{-1} - b^{-1}}$

64. $\left(\dfrac{a}{b}\right)^{-1} + \left(\dfrac{b}{a}\right)^{-1}$

65. Show that $\left(\dfrac{a}{b}\right)^{-n} = \left(\dfrac{b}{a}\right)^n$

Evaluate each expression in Exercises 66–69.

66. $(1.20^2)^{-1}$

67. $[(-3.67)^2]^{-1}$

68. $\left(\dfrac{7.65^{-1}}{7.65^2}\right)^2$

69. $\left(\dfrac{4.46^2}{4.46^{-1}}\right)^{-1}$

In Exercises 70–75, write each number using scientific notation.

70. 7000

71. 0.0091

72. 452,000,000,000

73. 23

74. 0.00000357

75. 0.8×10^{-3}

In Exercises 76–81, write each number without exponents.

76. 4.53×10^5

77. 8.93×10^{-4}

78. 0.0017×10^7

79. 145×10^3

80. 100×10^{-3}

81. 1253×10^{-6}

82. The dimensions of a rectangular field, measured in meters, are 4.1×10^3 by 3.75×10^5. Find the area of this field expressed in scientific notation.

83. The volume V of a spherical bubble of radius r is given by the formula

$$V = \frac{4}{3}\pi r^3$$

If we take the value of π to be 3.14, find the volume of a bubble, using scientific notation, if its radius is 0.09 inch.

84. Find the distance, expressed in scientific notation, that light travels in 0.000020 seconds if the speed of light is 1.86×10^5 miles per second.

85. The Republic of Singapore is said to have the highest population density of any country in the world. If its area is 270 square miles and its estimated population is 4,500,000, find the population density, that is, the approximate number of people per square mile, using scientific notation.

86. Scientists have suggested that the relationship between an animal's weight W and its surface area S is given by the formula

$$S = KW^{\frac{2}{3}}$$

where K is a constant chosen so that W, measured in kilograms, yields a value for S, measured in square meters. If the value of K for a horse is 0.10 and the horse weights 350 kilograms, find the estimated surface area of the horse, using scientific notation.

87. Simplify the following:

a. $\dfrac{2^{n+3} + 2^n + 2^n}{4(2^{n+3} - 2^{n+1})}$

b. $\dfrac{a(1 - r^3)}{1 - r}$

c. $\dfrac{9^{6m}}{3^{2m}}$

d. $\dfrac{8^4 + 8^4 + 8^4 + 8^4}{4^4}$

e. $\dfrac{(6 \times 10^{-2})(2 \times 10^{-3})}{3 \times 10^8}$

88. Assuming a lifetime is 70 years, how much is that in seconds? Express your answer in scientific notation.

1.7 Rational Exponents and Radicals

*n*th Roots

Consider a square whose area is 25 cm^2 (square centimeters), and whose sides are of length a. We can then write

$$a^2 = 25$$

so that a is a number whose square is 25. We say that a is the *square root* of b if $a^2 = b$. Similarly, we say that a is a *cube root* of b if $a^3 = b$; and, in general, if n is a natural number, we say that

a is an ***n*th root** of b if $a^n = b$

Thus, 5 is a square root of 25 since $5^2 = 25$, and -2 is a cube root of -8 since $(-2)^3 = -8$.

Since $(-5)^2 = 25$, we conclude that -5 is also a square root of 25. More generally, if $b > 0$ and a is a square root of b, then $-a$ is also a square root of b. If $b < 0$, there is no real number a such that $a^2 = b$, since the square of a real number is always nonnegative. (In Section 1.8, we introduce an extended number system in which there is a root when $b < 0$ and n is even.)

We would like to define rational exponents in a manner that is consistent with the rules for integer exponents. If the rule $(a^m)^n = a^{mn}$ is to hold, then we must have

$$(b^{1/n})^n = b^{n/n} = b$$

But a is an nth root of b if $a^n = b$. Then for every natural number n, we say that

$b^{1/n}$ is an nth root of b

Principal *n*th Root

If n is even and b is positive, there are two numbers that are nth roots of b. For example,

$$4^2 = 16 \quad \text{and} \quad (-4)^2 = 16$$

There are then two candidates for $16^{1/2}$, namely 4 and -4. To avoid ambiguity we say that $16^{1/2} = 4$ That is, if n is even and b is positive, we always *choose the positive number* a such that $a^n = b$ is the nth root, and call it the *principal* n*th root* of b. Thus, $b^{1/n}$ denotes the principal nth root of b.

We summarize these results in Table 11.

TABLE 11 Properties of Powers $a^n = b$ and Roots $a = b^{1/n}$ for Integer $n > 0$

Example		Property
$2^3 = 8$	$(-2)^3 = -8$	Any power of a real number is a real number.
$8^{1/3} = 2$	$(-8)^{1/3} = -2$	The odd root of a real number is a real number.
$0^n = 0$	$0^{1/n} = 0$	A positive power or root of zero is zero.
$4^2 = 16$	$(-4)^2 = 16$	A positive number raised to an even power equals the negative of that number raised to the same even power.
$(16)^{1/2} = 4$		The principal root of a positive number is a positive number.
$(-4)^{1/2}$ is undefined in the real number system.		The even root of a negative number is not a real number.

EXAMPLE 1 ROOTS OF A REAL NUMBER

Evaluate.

a. $144^{1/2}$

b. $(-8)^{1/3}$

c. $(-25)^{1/2}$

d. $-\left(\dfrac{1}{16}\right)^{1/4}$

SOLUTION

a. $144^{1/2} = 12$

b. $(-8)^{1/3} = -2$

c. $(-25)^{1/2}$ is not a real number

d. $-\left(\dfrac{1}{16}\right)^{1/4} = -\dfrac{1}{2}$

Rational Exponents

Now we are prepared to define $b^{m/n}$. Where m is an integer (positive or negative), n is a positive integer, and $b > 0$ when n is even. We want the rules for exponents to hold for rational exponents as well. That is, we want to have

$$4^{3/2} = 4^{(1/2)(3)} = (4^{1/2})^3 = 2^3 = 8$$

and

$$4^{3/2} = 4^{(3)(1/2)} = (4^3)^{1/2} = (64)^{1/2} = 8$$

To achieve this consistency, we define $b^{m/n}$ for an integer m, a natural number n, and a real number b, by

$$b^{m/n} = (b^{1/n})^m = (b^m)^{1/n}$$

where b must be positive when n is even. With this definition, all the rules of exponents continue to hold when the exponents are rational numbers.

EXAMPLE 2 OPERATIONS WITH RATIONAL EXPONENTS

Simplify.

a. $(-8)^{4/3}$ b. $x^{1/2} \cdot x^{3/4}$ c. $(x^{3/4})^2$ d. $(3x^{2/3}y^{-5/3})^3$

SOLUTION

a. $(-8)^{4/3} = [(-8)^{1/3}]^4 = (-2)^4 = 16$

b. $x^{1/2} \cdot x^{3/4} = x^{1/2+3/4} = x^{5/4}$

c. $(x^{3/4})^2 = x^{(3/4)(2)} = x^{3/2}$

d. $(3x^{2/3}y^{-5/3})^3 = 3^3 \cdot x^{(2/3)(3)}y^{(-5/3)(3)} = 27x^2y^{-5} = \dfrac{27x^2}{y^5}$

Focus: When Is a Proof Not a Proof?

Books of mathematical puzzles love to include “proofs” that lead to false or contradictory results. Of course, there is always an incorrect step hidden somewhere in the proof. The error may be subtle, but a good grounding in the fundamentals of mathematics will enable you to catch it.

Examine the following “proof.”

$$
\begin{aligned}
1 &= 1^{1/2} && (1)\\
&= [(-1)^2]^{1/2} && (2)\\
&= (-1)^{2/2} && (3)\\
&= (-1)^1 && (4)\\
&= -1 && (5)
\end{aligned}
$$

The result is obviously contradictory: we can’t have $1 = -1$. Yet each step seems to be legitimate. Did you spot the flaw? The rule

$$(b^m)^{1/n} = b^{m/n}$$

used in going from Equation (2) to (3) does not apply when n is even and b is negative.

✔ Progress Check

Simplify. Assume all variables are positive real numbers.

a. $27^{4/3}$ b. $(a^{1/2}b^{-2})^{-2}$ c. $\left(\dfrac{x^{1/3}y^{2/3}}{z^{5/6}}\right)^{12}$

Answers

a. 81 b. $\dfrac{b^4}{a}$ c. $\dfrac{x^4y^8}{z^{10}}$

Radicals

The symbol $\sqrt{b}$ is an alternative way of writing $b^{1/2}$; that is, $\sqrt{b}$ denotes the nonnegative square root of b. The symbol $\sqrt{\ }$ is called a **radical sign**, and $\sqrt{b}$ is called the **principal square root** of b. Thus,

$$\sqrt{25} = 5 \qquad \sqrt{0} = 0 \qquad \sqrt{-25} \text{ is undefined}$$

In general, the symbol $\sqrt[n]{b}$ is an alternative way of writing $b^{1/n}$, the principal nth root of b. Of course, we must apply the same restrictions to $\sqrt[n]{b}$ that we established for b$^{1/n}$. In summary:

$$\sqrt[n]{b} = b^{1/n} = a \qquad \text{where } a^n = b$$

with these restrictions:

- if n is even and $b < 0$, $\sqrt[n]{b}$ is not a real number;
- if n is even and $b \geq 0$, $\sqrt[n]{b}$ is the *nonnegative* number a satisfying $a^n = b$.

WARNING

Many students are accustomed to writing $\sqrt{4} = \pm 2$. This is incorrect since the symbol $\sqrt{\ }$ indicates the *principal* square root, which is nonnegative. Get in the habit of writing $\sqrt{4} = 2$. If you want to indicate *all* square roots of 4, write $\pm\sqrt{4} = \pm 2$.

In short, $\sqrt[n]{b}$ is the **radical form of** $b^{1/n}$. We can switch back and forth from one form to the other. For instance,

$$\sqrt[3]{7} = 7^{1/3} \qquad (11)^{1/5} = \sqrt[5]{11}$$

Finally, we treat the radical form of $b^{m/n}$ where m is an integer and n is a positive integer as follows:

$$b^{m/n} = (b^m)^{1/n} = \sqrt[n]{b^m}$$

and

$$b^{m/n} = (b^{1/n})^m = (\sqrt[n]{b})^m$$

Thus

$$\begin{aligned} 8^{2/3} &= (8^2)^{1/3} = \sqrt[3]{8^2} \\ &= (8^{1/3})^2 = (\sqrt[3]{8})^2 \end{aligned}$$

(Check that the last two expressions have the same value.)

EXAMPLE 3 RADICALS AND RATIONAL EXPONENTS

Change from radical form to rational exponent form or vice versa. Assume all variables are nonzero.

a. $(2x)^{-3/2}, \quad x > 0$

b. $\dfrac{1}{\sqrt[7]{y^4}}$

c. $(-3a)^{3/7}$

d. $\sqrt{x^2 + y^2}$

SOLUTION

a. $(2x)^{-3/2} = \dfrac{1}{(2x)^{3/2}} = \dfrac{1}{\sqrt{8x^3}}$

b. $\dfrac{1}{\sqrt[7]{y^4}} = \dfrac{1}{y^{4/7}} = y^{-4/7}$

c. $(-3a)^{3/7} = \sqrt[7]{-27a^3}$

d. $\sqrt{x^2 + y^2} = (x^2 + y^2)^{1/2}$

✔ Progress Check

Change from radical form to rational exponent form or vice versa. Assume all variables are positive real numbers.

a. $\sqrt[4]{2rs^3}$ b. $(x + y)^{5/2}$ c. $y^{-5/4}$ d. $\dfrac{1}{\sqrt[4]{m^5}}$

Answers

a. $(2r)^{1/4}s^{3/4}$ b. $\sqrt{(x + y)^5}$ c. $\dfrac{1}{\sqrt[4]{y^5}}$ d. $m^{-5/4}$

Since radicals are just another way of writing exponents, the properties of radicals can be derived from the properties of exponents. In Table 12, n is a positive integer, a and b are real numbers, and all radicals are real numbers.

TABLE 12 Properties of Radicals

Example	Property
$\sqrt[3]{8^2} = (\sqrt[3]{8})^2 = 4$	$\sqrt[n]{b^m} = (\sqrt[n]{b})^m$
$\sqrt{4}\sqrt{9} = \sqrt{36} = 6$	$\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}$
$\dfrac{\sqrt[3]{8}}{\sqrt[3]{27}} = \sqrt[3]{\dfrac{8}{37}} = \dfrac{2}{3}$	$\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$
$\sqrt[3]{(-2)^3} = -2$	$\sqrt[n]{a^n} = a$ if n is odd
$\sqrt{(-2)^2} = \|-2\| = 2$	$\sqrt[n]{a^n} = \|a\|$ if n is even

Here are some examples using these properties.

EXAMPLE 4 OPERATIONS WITH RADICALS

Simplify.

a. $\sqrt{18}$ b. $\sqrt[3]{-54}$ c. $2\sqrt[3]{8x^3y}$ d. $\sqrt{x^6}$

SOLUTION

a. $\sqrt{18} = \sqrt{9 \cdot 2} = \sqrt{9}\sqrt{2} = 3\sqrt{2}$

b. $\sqrt[3]{-54} = \sqrt[3]{(-27)(2)} = \sqrt[3]{-27}\sqrt[3]{2} = -3\sqrt[3]{2}$

c. $2\sqrt[3]{8x^3y} = 2\sqrt[3]{8}\sqrt[3]{x^3}\sqrt[3]{y} = 2(2)(x)\sqrt[3]{y} = 4x\sqrt[3]{y}$

d. $\sqrt{x^6} = \sqrt{x^2} \cdot \sqrt{x^2} \cdot \sqrt{x^2} = |x| \cdot |x| \cdot |x| = |x|^3$

WARNING

The properties of radicals state that

$$\sqrt{x^2} = |x|$$

It is a common error to write $\sqrt{x^2} = x$. This can lead to the conclusion that $\sqrt{(-6)^2} = -6$. Since the symbol $\sqrt{\ }$ represents the principal, or nonnegative, square root of a number, the result cannot be negative. It is therefore essential to write $\sqrt{x^2} = |x|$ (and, in fact, $\sqrt[n]{x^n} = |x|$ whenever n is even) unless we know that $x \geq 0$, in which case we can write $\sqrt{x^2} = x$.

Simplifying Radicals

A radical is said to be in **simplified form** when the following conditions are satisfied:

1. $\sqrt[n]{b^m}$ has $m < n$;

2. $\sqrt[n]{b^m}$ has no common factors between m and n;

3. A denominator is free of radicals.

The first two conditions can always be met by using the properties of radicals and by writing radicals in exponent form. For example,

$$\sqrt[3]{x^4} = \sqrt[3]{x^3 \cdot x} = \sqrt[3]{x^3}\sqrt[3]{x} = x\sqrt[3]{x}$$

and

$$\sqrt[6]{x^4} = x^{4/6} = x^{2/3} = \sqrt[3]{x^2}$$

The third condition can always be satisfied by multiplying the fraction by a properly chosen form of unity, a process called **rationalizing the denominator**. For example, to rationalize $\frac{1}{\sqrt{3}}$ we proceed as follows:

$$\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{\sqrt{3^2}} = \frac{\sqrt{3}}{3}$$

In this connection, a useful formula is

$$(\sqrt{m} + \sqrt{n})(\sqrt{m} - \sqrt{n}) = m - n$$

which we will apply in the following examples.

EXAMPLE 5 RATIONALIZING DENOMINATORS

Rationalize the denominator. Assume all variables denote positive numbers.

a. $\sqrt{\dfrac{x}{y}}$ b. $\dfrac{4}{\sqrt{5} - \sqrt{2}}$ c. $\dfrac{5}{\sqrt{x} + 2}$ d. $\dfrac{5}{\sqrt{x + 2}}$

SOLUTION

a. $\sqrt{\dfrac{x}{y}} = \dfrac{\sqrt{x}}{\sqrt{y}} = \dfrac{\sqrt{x}}{\sqrt{y}} \cdot \dfrac{\sqrt{y}}{\sqrt{y}} = \dfrac{\sqrt{xy}}{\sqrt{y^2}} = \dfrac{\sqrt{xy}}{y}$

b. $\dfrac{4}{\sqrt{5}-\sqrt{2}} = \dfrac{4}{\sqrt{5}-\sqrt{2}} \cdot \dfrac{\sqrt{5} + \sqrt{2}}{\sqrt{5} + \sqrt{2}} = \dfrac{4(\sqrt{5} + \sqrt{2})}{5-2} = \dfrac{4}{3}(\sqrt{5} + \sqrt{2})$

c. $\dfrac{5}{\sqrt{x} + 2} = \dfrac{5}{\sqrt{x} + 2} \cdot \dfrac{\sqrt{x}-2}{\sqrt{x}-2} = \dfrac{5(\sqrt{x} - 2)}{x - 4}$

d. $\dfrac{5}{\sqrt{x + 2}} = \dfrac{5}{\sqrt{x + 2}} \cdot \dfrac{\sqrt{x + 2}}{\sqrt{x + 2}} = \dfrac{5\sqrt{x + 2}}{x + 2}$

✔ Progress Check

Rationalize the denominator. Assume all variables denote positive numbers.

a. $\dfrac{-9xy^3}{\sqrt{3xy}}$ b. $\dfrac{-6}{\sqrt{2}+\sqrt{6}}$ c. $\dfrac{4}{\sqrt{x}-\sqrt{y}}$

Answers

a. $-3y^2\sqrt{3xy}$ b. $\dfrac{3}{2}(\sqrt{2}-\sqrt{6})$ c. $\dfrac{4(\sqrt{x}+\sqrt{y})}{x-y}$

There are times in mathematics when it is necessary to rationalize the numerator instead of the denominator. Although this is in opposition to a simplified form, we illustrate this technique with the following example. Note that if an expression does not display a denominator, we assume a denominator of 1.

EXAMPLE 6 RATIONALIZING NUMERATORS

Rationalize the numerator. Assume all variables denote positive numbers.

a. $\dfrac{4}{3}(\sqrt{5}+\sqrt{2})$ b. $\dfrac{x-\sqrt{3}}{x+4}$

c. $\sqrt{x}+4$ d. $\dfrac{\sqrt{x}-2}{x-4}$

SOLUTION

a. $\dfrac{4}{3}(\sqrt{5}+\sqrt{2})\cdot\dfrac{\sqrt{5}-\sqrt{2}}{\sqrt{5}-\sqrt{2}}=\dfrac{4}{3}\,\dfrac{(5-2)}{(\sqrt{5}-\sqrt{2})}=\dfrac{4}{\sqrt{5}-\sqrt{2}}$

See Example 5 (b).

b. $\dfrac{x-\sqrt{3}}{x+4}\cdot\dfrac{x+\sqrt{3}}{x+\sqrt{3}}=\dfrac{x^2-3}{(x+4)(x+\sqrt{3})}$

c. $\dfrac{\sqrt{x}+4}{1}\cdot\dfrac{\sqrt{x}-4}{\sqrt{x}-4}=\dfrac{x-16}{\sqrt{x}-4}$

d. $\dfrac{\sqrt{x}-2}{x-4}\cdot\dfrac{\sqrt{x}+2}{\sqrt{x}+2}=\dfrac{x-4}{(x-4)(\sqrt{x}+2)}=\dfrac{1}{\sqrt{x}+2},\qquad x\neq 4$

EXAMPLE 7 SIMPLIFIED FORMS WITH RADICALS

Write in simplified form. Assume all variables denote positive numbers.

a. $\sqrt[4]{y^5}$ b. $\sqrt{\dfrac{8x^3}{y}}$ c. $\sqrt[6]{\dfrac{x^3}{y^2}}$

SOLUTION

a. $\sqrt[4]{y^5} = \sqrt[4]{y^4 \cdot y} = \sqrt[4]{y^4}\sqrt[4]{y} = y\sqrt[4]{y}$

b.
$$\sqrt{\frac{8x^3}{y}} = \frac{\sqrt{(4x^2)(2x)}}{\sqrt{y}} = \frac{\sqrt{4x^2}\sqrt{2x}}{\sqrt{y}} = \frac{2x\sqrt{2x}}{\sqrt{y}} = \frac{2x\sqrt{2x}}{\sqrt{y}} \cdot \frac{\sqrt{y}}{\sqrt{y}} = \frac{2x\sqrt{2xy}}{y}$$

c. $\sqrt[6]{\dfrac{x^3}{y^2}} = \dfrac{\sqrt[6]{x^3}}{\sqrt[6]{y^2}} = \dfrac{\sqrt{x}}{\sqrt[3]{y}} = \dfrac{\sqrt{x}}{\sqrt[3]{y}} \cdot \dfrac{\sqrt[3]{y^2}}{\sqrt[3]{y^2}} = \dfrac{\sqrt{x}\sqrt[3]{y^2}}{y}$

✔ Progress Check

Write in simplified form. Assume all variables denote positive numbers.

a. $\sqrt{75}$ b. $\sqrt{\dfrac{18x^6}{y}}$ c. $\sqrt[3]{ab^4c^7}$ d. $\dfrac{-2xy^3}{\sqrt[4]{32x^3y^5}}$

Answers

a. $5\sqrt{3}$ b. $\dfrac{3|x|^3\sqrt{2y}}{y}$ c. $bc^2\sqrt[3]{abc}$ d. $-\dfrac{y}{2}\sqrt[4]{8xy^3}$

Operations with Radicals

We can add or subtract expressions involving exactly the same radical forms. For example,

$$2\sqrt{2} + 3\sqrt{2} = 5\sqrt{2}$$

since

$$2\sqrt{2} + 3\sqrt{2} = (2 + 3)\sqrt{2} = 5\sqrt{2}$$

and

$$3\sqrt[3]{x^2y} - 7\sqrt[3]{x^2y} = -4\sqrt[3]{x^2y}$$

EXAMPLE 8 ADDITION AND SUBTRACTION OF RADICALS

Write in simplified form. Assume all variables denote positive numbers.

a. $7\sqrt{5} + 4\sqrt{3} - 9\sqrt{5}$

b. $\sqrt[3]{x^2y} - \dfrac{1}{2}\sqrt{xy} - 3\sqrt[3]{x^2y} + 4\sqrt{xy}$

SOLUTION

a. $7\sqrt{5} + 4\sqrt{3} - 9\sqrt{5} = -2\sqrt{5} + 4\sqrt{3}$

b. $\sqrt[3]{x^2y} - \dfrac{1}{2}\sqrt{xy} - 3\sqrt[3]{x^2y} + 4\sqrt{xy} = -2\sqrt[3]{x^2y} + \dfrac{7}{2}\sqrt{xy}$

WARNING

$$\sqrt{9} + \sqrt{16} \neq \sqrt{25}$$

You can perform addition only with identical radical forms. *Adding unlike radicals is one of the most common mistakes made by students in algebra!* You can easily verify that

$$\sqrt{9} + \sqrt{16} = 3 + 4 = 7$$

The product of $\sqrt[n]{a}$ and $\sqrt[m]{b}$ can be readily simplified only when $m = n$. Thus,

$$\sqrt[5]{x^2y} \cdot \sqrt[5]{xy} = \sqrt[5]{x^3y^2}$$

but

$$\sqrt[3]{x^2y} \cdot \sqrt[5]{xy}$$

cannot be readily simplified.

EXAMPLE 9 MULTIPLICATION OF RADICALS

Multiply and simplify.

a. $2\sqrt[3]{xy^2} \cdot \sqrt[3]{x^2y^2}$ b. $\sqrt[5]{a^2b}\sqrt[5]{ab}\sqrt[5]{ab^2}$

SOLUTION

a. $2\sqrt[3]{xy^2} \cdot \sqrt[3]{x^2y^2} = 2\sqrt[3]{x^3y^4} = 2xy\sqrt[3]{y}$

b. $\sqrt[5]{a^2b}\sqrt[5]{ab}\sqrt[5]{ab^2} = \sqrt[5]{a^3b^3}\sqrt[5]{ab}$

Calculator Alert

Most calculators have a $\boxed{\sqrt{\ }}$ key. Scientific and graphing calculators sometimes have a special key to evaluate other roots. This key may be $\boxed{\sqrt[x]{y}}$, $\boxed{\sqrt[x]{\ }}$, or $\boxed{x^{1/y}}$. If your calculator does not have a special root key, you can use the power key to evaluate roots.

Example: Show that $\sqrt[5]{12} = 1.64375183$.

Solution: If your calculator has a root key, evaluate $5\boxed{\sqrt[x]{y}}12$. Otherwise, evaluate $12\boxed{x^y}(1 \div 5)$ or $12\boxed{\wedge}(1 \div 5)$ on your calculator.

Exercise Set 1.7

In Exercises 1–12, simplify, and write the answer using only positive exponents.

1. $16^{3/4}$
2. $(-125)^{-1/3}$
3. $(-64)^{-2/3}$
4. $c^{1/4}c^{-2/3}$
5. $\dfrac{2x^{1/3}}{x^{-3/4}}$
6. $\dfrac{y^{-2/3}}{y^{1/5}}$
7. $\left(\dfrac{x^{3/2}}{x^{2/3}}\right)^{1/6}$
8. $\dfrac{125^{4/3}}{125^{2/3}}$
9. $(x^{1/3}y^2)^6$
10. $(x^6y^4)^{-1/2}$
11. $\left(\dfrac{x^{15}}{y^{10}}\right)^{3/5}$
12. $\left(\dfrac{x^{18}}{y^{-6}}\right)^{2/3}$

In Exercises 13–18, write the expression in radical form.

13. $\left(\dfrac{1}{4}\right)^{2/5}$
14. $x^{2/3}$
15. $a^{3/4}$
16. $(-8x^2)^{2/5}$
17. $(12x^3y^{-2})^{2/3}$
18. $\left(\dfrac{8}{3}x^{-2}y^{-4}\right)^{-3/2}$

In Exercises 19–24, write the expression in exponent form.

19. $\sqrt[4]{8^3}$
20. $\sqrt[5]{3^2}$
21. $\dfrac{1}{\sqrt[5]{(-8)^2}}$
22. $\dfrac{1}{\sqrt[3]{x^7}}$
23. $\dfrac{1}{\sqrt[4]{\dfrac{4}{9}a^3}}$
24. $\sqrt[5]{(2a^2b^3)^4}$

In Exercises 25–33, evaluate the expression. Verify your answer using your calculator.

25. $\sqrt{\dfrac{4}{9}}$
26. $\sqrt{\dfrac{25}{4}}$
27. $\sqrt[4]{-81}$
28. $\sqrt[3]{\dfrac{1}{27}}$
29. $\sqrt{(5)^2}$
30. $\sqrt{\left(\dfrac{-1}{3}\right)^2}$
31. $\sqrt{\left(\dfrac{5}{4}\right)^2}$
32. $\sqrt{\left(-\dfrac{7}{2}\right)^2}$
33. $(14.43)^{3/2}$

In Exercises 34–36, provide a real value for each variable to demonstrate the result.

34. $\sqrt{x^2} \neq x$
35. $\sqrt{x^2 + y^2} \neq x + y$
36. $\sqrt{x}\sqrt{y} \neq xy$

In Exercises 37–56, write the expression in simplified form. Every variable represents a positive number.

37. $\sqrt{48}$
38. $\sqrt{200}$
39. $\sqrt[3]{54}$
40. $\sqrt{x^8}$
41. $\sqrt[3]{y^7}$
42. $\sqrt[4]{b^{14}}$
43. $\sqrt[4]{96x^{10}}$
44. $\sqrt{x^5y^4}$
45. $\sqrt{x^5y^3}$
46. $\sqrt[3]{24b^{10}c^{14}}$
47. $\sqrt[4]{16x^8y^5}$
48. $\sqrt{20x^5y^7z^4}$
49. $\sqrt{\dfrac{1}{5}}$
50. $\dfrac{4}{3\sqrt{11}}$
51. $\dfrac{1}{\sqrt{3y}}$
52. $\sqrt{\dfrac{2}{y}}$
53. $\dfrac{4x^2}{\sqrt{2x}}$
54. $\dfrac{8a^2b^2}{2\sqrt{2b}}$
55. $\sqrt[3]{x^2y^7}$
56. $\sqrt[4]{48x^8y^6z^2}$

In Exercises 57–66, simplify and combine terms.

57. $2\sqrt{3} + 5\sqrt{3}$
58. $4\sqrt[3]{11} - 6\sqrt[3]{11}$
59. $3\sqrt{x} + 4\sqrt{x}$
60. $3\sqrt{2} + 5\sqrt{2} - 2\sqrt{2}$
61. $2\sqrt{27} + \sqrt{12} - \sqrt{48}$
62. $\sqrt{20} - 4\sqrt{45} + \sqrt{80}$
63. $\sqrt[3]{40} + \sqrt{45} - \sqrt[3]{135} + 2\sqrt{80}$
64. $\sqrt{2abc} - 3\sqrt{8abc} + \sqrt{\dfrac{abc}{2}}$
65. $2\sqrt{5} - (3\sqrt{5} + 4\sqrt{5})$
66. $2\sqrt{18} - (3\sqrt{12} - 2\sqrt{75})$

In Exercises 67–74, multiply and simplify.

67. $\sqrt{3}(\sqrt{3} + 4)$
68. $\sqrt{8}(\sqrt{2} - \sqrt{3})$
69. $3\sqrt[3]{x^2y}\sqrt[3]{xy^2}$

70. $-4\sqrt[5]{x^2y^3}\sqrt[5]{x^4y^2}$

71. $(\sqrt{2} - \sqrt{3})^2$

72. $(\sqrt{8} - 2\sqrt{2})(\sqrt{2} + 2\sqrt{8})$

73. $(\sqrt{3x} + \sqrt{2y})(\sqrt{3x} - 2\sqrt{2y})$

74. $(\sqrt[3]{2x} + 3)(\sqrt[3]{2x} - 3)$

In Exercises 75–86, rationalize the denominator.

75. $\dfrac{3}{\sqrt{2} + 3}$

76. $\dfrac{-3}{\sqrt{7}-9}$

77. $\dfrac{-2}{\sqrt{3}-4}$

78. $\dfrac{3}{\sqrt{x}-5}$

79. $\dfrac{-3}{3\sqrt{a} + 1}$

80. $\dfrac{4}{2-\sqrt{2y}}$

81. $\dfrac{-3}{5 + \sqrt{5y}}$

82. $\dfrac{\sqrt{3}}{\sqrt{3}-5}$

83. $\dfrac{\sqrt{2} + 1}{\sqrt{2} - 1}$

84. $\dfrac{\sqrt{5} + \sqrt{3}}{\sqrt{5}-\sqrt{3}}$

85. $\dfrac{\sqrt{6} + \sqrt{2}}{\sqrt{3} - \sqrt{2}}$

86. $\dfrac{2\sqrt{a}}{\sqrt{2x} + \sqrt{y}}$

In Exercises 87–90, rationalize the numerator.

87. $\sqrt{12} - \sqrt{10}$

88. $\dfrac{3-\sqrt{x}}{x-9}$

89. $\dfrac{\sqrt{x}-4}{16-x}$

90. $\dfrac{2-\sqrt{x + 1}}{3-x}$

In Exercises 91 and 92, provide real values for x and y and a positive integer value for n to demonstrate the result.

91. $\sqrt{x} + \sqrt{y} \neq \sqrt{x + y}$

92. $\sqrt[n]{x^n + y^n} \neq x + y$

93. Find the step in the following "proof" that is incorrect. Explain.

$$2 = \sqrt{4} = \sqrt{(-2)(-2)} = \sqrt{-2}\sqrt{-2} = -2$$

94. Prove that $|ab| = |a||b|$. (*Hint:* Begin with $|ab| = \sqrt{(ab)^2}$.)

95. Simplify the following.

a. $\sqrt{x\sqrt{x\sqrt{x}}}$

b. $(x^{1/2} - x^{-1/2})^2$

c. $\sqrt{1 + x^2} - \dfrac{\sqrt{1 + x^2}}{2}$

d. $\sqrt[5]{\dfrac{3^4 + 3^4 + 3^4}{5^4 + 5^4 + 5^4 + 5^4 + 5^4}}$

e. $\dfrac{5(1 + x^2)^{1/2} - 5x^2(1 + x^2)^{-1/2}}{1 + x^2}$

96. Write the following in simplest radical form.

a. $\sqrt{a^{-2} + c^{-2}}$

b. $\sqrt{1-\left(\dfrac{a}{c}\right)^2}$

c. $\sqrt{x + \dfrac{1}{x} + 2}$

97. The frequency of an electrical circuit is given by

$$\frac{1}{2\pi}\sqrt{\frac{Lc_1c_2}{c_1 + c_2}}$$

Make the denominator radical free. (***Hint:*** Use the techniques for rationalizing the denominator.)

98. Use your calculator to find $\sqrt{0.4}$, $\sqrt{0.04}$, $\sqrt{0.004}$, $\sqrt{0.0004}$, and so on, until you see a pattern. Can you state a rule about the value of

$$\sqrt{\frac{a}{10^n}}$$

where a is a perfect square and n is a positive integer? Under what circumstances does this expression have an integer value? Test your rule for large values of n.

1.8 Complex Numbers

One of the central problems in algebra is to find solutions to a given polynomial equation. This problem will be discussed in later chapters of this book. For now, observe that there is no real number that satisfies a polynomial equation such as

$$x^2 = -4$$

since the square of a real number is always nonnegative.

To resolve this problem, mathematicians created a new number system built upon an **imaginary unit *i***, defined by $i = \sqrt{-1}$. This number i has the property that when we square both sides of the equation we have $i^2 = -1$, a result that cannot be obtained with real numbers. By definition,

$$i = \sqrt{-1}$$
$$i^2 = -1$$

We also assume that i behaves according to all the algebraic laws we have already developed (with the exception of the rules for inequalities for real numbers). This allows us to simplify higher powers of i. Thus,

$$i^3 = i^2 \cdot i = (-1)i = -i$$
$$i^4 = i^2 \cdot i^2 = (-1)(-1) = 1$$

Now we may simplify i^n when n is any natural number. Since $i^4 = 1$, we seek the highest multiple of 4 that is less than or equal to n. For example,

$$i^5 = i^4 \cdot i = (1) \cdot i = i$$
$$i^{27} = i^{24} \cdot i^3 = (i^4)^6 \cdot i^3 = (1)^6 \cdot i^3 = i^3 = -i$$

EXAMPLE 1 IMAGINARY UNIT *i*

Simplify.

a. i^{51} b. $-i^{74}$

SOLUTION

a. $i^{51} = i^{48} \cdot i^3 = (i^4)^{12} \cdot i^3 = (1)^{12} \cdot i^3 = i^3 = -i$

b. $-i^{74} = -i^{72} \cdot i^2 = -(i^4)^{18} \cdot i^2 = -(1)^{18} \cdot i^2 = -(1)(-1) = 1$

We may also write square roots of negative numbers in terms of i. For example,

$$\sqrt{-25} = i\sqrt{25} = 5i$$

and, in general, we define

$$\sqrt{-a} = i\sqrt{a} \qquad \text{for } a > 0$$

Any number of the form bi, where b is a real number, is called an **imaginary number**.

WARNING

$$\sqrt{-4}\sqrt{-9} \neq \sqrt{36}$$

The rule $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$ holds only when $a \geq 0$ and $b \geq 0$. Instead, write

$$\sqrt{-4}\sqrt{-9} = 2i \cdot 3i = 6i^2 = -6$$

Having created imaginary numbers, we next combine real and imaginary numbers. We say that $a + bi$ is a **complex number** where a and b are real numbers. The number a is called the **real part** of $a + bi$, and b is called the **imaginary part**. The following are examples of complex numbers.

$$3 + 2i \qquad 2 - i \qquad -2i \qquad \frac{4}{5} + \frac{1}{5}i$$

Note that every real number a can be written as a complex number by choosing $b = 0$. Thus,

$$a = a + 0i$$

We see that the real number system is a subset of the complex number system. The desire to find solutions to every quadratic equation has led mathematicians to create a more comprehensive number system, which incorporates all previous number systems. We will show in a later chapter that complex numbers are all that we need to provide solutions to any polynomial equation.

EXAMPLE 2 COMPLEX NUMBERS $a + bi$

Write as a complex number:

a. $-\frac{1}{2}$ b. $\sqrt{-9}$ c. $-1 - \sqrt{-4}$

SOLUTION

a. $-\frac{1}{2} = -\frac{1}{2} + 0i$

b. $\sqrt{-9} = i\sqrt{9} = 3i = 0 + 3i$

c. $-1 - \sqrt{-4} = -1 - i\sqrt{4} = -1 - 2i$

Do not be concerned by the word “complex.” You already have all the basic tools you need to tackle this number system. We will next define operations with complex numbers in such a way that the rules for the real numbers and the imaginary unit i continue to hold. We begin with equality and say that two complex numbers are **equal** if their real parts are equal and their imaginary parts are equal; that is,

$$a + bi = c + di \quad \text{if} \quad a = c \quad \text{and} \quad b = d$$

EXAMPLE 3 EQUALITY OF COMPLEX NUMBERS

Solve the equation $x + 3i = 6 - yi$ for x and y.

SOLUTION

Equating the real parts, we have $x = 6$; equating the imaginary parts, $3 = -y$ or $y = -3$.

Complex numbers are added and subtracted by adding or subtracting the real parts and by adding or subtracting the imaginary parts.

Addition and Subtraction of Complex Numbers

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

Note that the sum or difference of two complex numbers is again a complex number.

EXAMPLE 4 ADDITION AND SUBTRACTION OF COMPLEX NUMBERS

Perform the indicated operations.

a. $(7 - 2i) + (4 - 3i)$ b. $14 - (3 - 8i)$

SOLUTION

a. $(7 - 2i) + (4 - 3i) = (7 + 4) + (-2 - 3)i = 11 - 5i$

b. $14 - (3 - 8i) = (14 - 3) + 8i = 11 + 8i$

✔ Progress Check

Perform the indicated operations.

a. $(-9 + 3i) + (6 - 2i)$ b. $7i - (3 + 9i)$

Answers

a. $-3 + i$ b. $-3 - 2i$

We now define multiplication of complex numbers in a manner that permits the commutative, associative, and distributive laws to hold, along with the definition $i^2 = -1$. We must have

$$\begin{aligned}(a + bi)(c + di) &= a(c + di) + bi(c + di)\\ &= ac + adi + bci + bdi^2\\ &= ac + (ad + bc)i + bd(-1)\\ &= (ac - bd) + (ad + bc)i\end{aligned}$$

The rule for multiplication is

Multiplication of Complex Numbers

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

This result demonstrates that the product of two complex numbers is a complex number. It need not be memorized. Use the distributive law to form all the products and the substitution $i^2 = -1$ to simplify.

EXAMPLE 5 MULTIPLICATION OF COMPLEX NUMBERS

Find the product of $(2 - 3i)$ and $(7 + 5i)$.

SOLUTION

$$\begin{aligned}(2 - 3i)(7 + 5i) &= 2(7 + 5i) - 3i(7 + 5i)\\ &= 14 + 10i - 21i - 15i^2\\ &= 14 - 11i - 15(-1)\\ &= 29 - 11i\end{aligned}$$

✔ Progress Check

Find the product.

a. $(-3 - i)(4 - 2i)$ b. $(-4 - 2i)(2 - 3i)$

Answers

a. $-14 + 2i$ b. $-14 + 8i$

The complex number $a - bi$ is called the **complex conjugate**, or simply the **conjugate**, of the complex number $a + bi$. For example, $3 - 2i$ is the conjugate of $3 + 2i$, $4i$ is the conjugate of $-4i$, and 2 is the conjugate of 2. Forming the product $(a + bi)(a - bi)$, we have

$$(a + bi)(a-bi) = a^2 - abi + abi - b^2i^2$$

$$= a^2 + b^2 \qquad \text{Since } i^2 = -1$$

Because a and b are real numbers, $a^2 + b^2$ is also a real number. We can summarize this result as follows:

The Complex Conjugate and Multiplication

The complex conjugate of $a + bi$ is $a - bi$. The product of a complex number and its conjugate is a real number.

$$(a + bi)(a - bi) = a^2 + b^2$$

Before we examine the quotient of two complex numbers, we consider the reciprocal of $a + bi$, namely, $\frac{1}{a + bi}$. This may be simplified by multiplying both numerator and denominator by the conjugate of the denominator.

$$\frac{1}{a + bi} = \left(\frac{1}{a + bi}\right)\left(\frac{a - bi}{a - bi}\right) = \frac{a - bi}{a^2 + b^2} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i$$

In general, the quotient of two complex numbers

$$\frac{a + bi}{c + di}$$

is simplified in a similar manner, that is, by multiplying both numerator and denominator by the conjugate of the denominator.

$$\begin{aligned}\frac{a + bi}{c + di} &= \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} \\ &= \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2} \\ &= \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i\end{aligned}$$

Division of Complex Numbers

$$\frac{a + bi}{c + di} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i, \qquad c^2 + d^2 \neq 0$$

This result demonstrates that the quotient of two complex numbers is a complex number. Instead of memorizing this formula for division, remember that quotients of complex numbers may be simplified by multiplying the numerator and denominator by the conjugate of the denominator.

EXAMPLE 6 DIVISION OF COMPLEX NUMBERS

a. Write the quotient $\frac{-2 + 3i}{3 - 2i}$ in the form $a + bi$.

b. Write the reciprocal of $2 - 5i$ in the form $a + bi$.

SOLUTION

a. Multiplying numerator and denominator by the conjugate of the denominator, $3 + 2i$, we have

$$\frac{-2 + 3i}{3 - 2i} = \frac{-2 + 3i}{3 - 2i} \cdot \frac{3 + 2i}{3 + 2i} = \frac{-6-4i + 9i + 6i^2}{3^2 + 2^2} = \frac{-6 + 5i + 6(-1)}{9 + 4}$$

$$= \frac{-12 + 5i}{13} = -\frac{12}{13} + \frac{5}{13}i$$

b. The reciprocal is $\frac{1}{2 - 5i}$. Multiplying both numerator and denominator by the conjugate $2 + 5i$, we have

$$\frac{1}{2 - 5i} \cdot \frac{2 + 5i}{2 + 5i} = \frac{2 + 5i}{2^2 + 5^2} = \frac{2 + 5i}{29} = \frac{2}{29} + \frac{5}{29}i$$

Verify that

$$(2 - 5i)\left(\frac{2}{29} + \frac{5}{29}i\right) = 1$$

✔ Progress Check

Write the following in the form $a + bi$.

a. $\frac{4 - 2i}{5 + 2i}$ b. $\frac{1}{2 - 3i}$ c. $\frac{-3i}{3 + 5i}$

Answers

a. $\frac{16}{29} - \frac{18}{29}i$ b. $\frac{2}{13} + \frac{3}{13}i$ c. $-\frac{15}{34} - \frac{9}{34}i$

Calculator Alert

Some scientific and graphing calculators have the capability to do computations with complex numbers. Consult your owner's manual for details. The owner's manual may be available online. Look up your calculator by model and number.

Exercise Set 1.8

Simplify in Exercises 1–9.

1. i^{60}
2. i^{27}
3. i^{83}
4. $-i^{54}$
5. $-i^{33}$
6. i^{-15}

7. i^{-84}

8. $-i^{39}$

9. $-i^{-25}$

In Exercises 10–21, write the number in the form $a + bi$.

10. 2

11. $-\frac{3}{4}$

12. -0.3

13. $\sqrt{-25}$

14. $-\sqrt{-5}$

15. $-\sqrt{-36}$

16. $-\sqrt{-18}$

17. $3 - \sqrt{-49}$

18. $-\frac{3}{2} - \sqrt{-72}$

19. $0.3 - \sqrt{-98}$

20. $-0.5 + \sqrt{-32}$

21. $-2 - \sqrt{-16}$

In Exercises 22–26, solve for x and y.

22. $(x + 2) + (2y - 1)i = -1 + 5i$

23. $(3x - 1) + (y + 5)i = 1 - 3i$

24. $\left(\frac{1}{2}x + 2\right) + (3y - 2)i = 4 - 7i$

25. $(2y + 1) - (2x - 1)i = -8 + 3i$

26. $(y - 2) + (5x - 3)i = 5$

In Exercises 27–42, compute the answer and write it in the form $a + bi$.

27. $2i + (3 - i)$

28. $-3i + (2 - 5i)$

29. $2 + 3i + (3 - 2i)$

30. $(3 - 2i) - \left(2 + \frac{1}{2}i\right)$

31. $-3 - 5i - (2 - i)$

32. $\left(\frac{1}{2} - i\right) + \left(1 - \frac{2}{3}i\right)$

33. $-2i(3 + i)$

34. $3i(2 - i)$

35. $i\left(-\frac{1}{2} + i\right)$

36. $\frac{i}{2}\left(\frac{4 - i}{2}\right)$

37. $(2 - i)(2 + i)$

38. $(5 + i)(2 - 3i)$

39. $(-2 - 2i)(-4 - 3i)$

40. $(2 + 5i)(1 - 3i)$

41. $(3 - 2i)(2 - i)$

42. $(4 - 3i)(2 + 3i)$

In Exercises 43–48, multiply by the conjugate and simplify.

43. $2 - i$

44. $3 + i$

45. $3 + 4i$

46. $2 - 3i$

47. $-4 - 2i$

48. $5 + 2i$

In Exercises 49–57, perform the indicated operations and write the answer in the form $a + bi$.

49. $\frac{2 + 5i}{1 - 3i}$

50. $\frac{1 + 3i}{2 - 5i}$

51. $\frac{3 - 4i}{3 + 4i}$

52. $\frac{4 - 3i}{4 + 3i}$

53. $\frac{3 - 2i}{2 - i}$

54. $\frac{2 - 3i}{3 - i}$

55. $\frac{2 + 5i}{3i}$

56. $\frac{5 - 2i}{-3i}$

57. $\frac{4i}{2 + i}$

In Exercises 58–64, find the reciprocal and write the answer in the form $a + bi$.

58. $3 + 2i$

59. $4 + 3i$

60. $\frac{1}{2} - i$

61. $1 - \frac{1}{3}i$

62. $-7i$

63. $-5i$

64. $\frac{3 - i}{3 + 2i}$

In Exercises 65–68, evaluate the polynomial $x^2 - 2x + 5$ for the given complex value of x.

65. $1 + 2i$

66. $2 - i$

67. $1 - i$

68. $1 - 2i$

69. Prove that the commutative law of addition holds for the set of complex numbers.

70. Prove that the commutative law of multiplication holds for the set of complex numbers.

71. Prove that $0 + 0i$ is the additive identity and $1 + 0i$ is the multiplicative identity for the set of complex numbers.

72. Prove that $-a - bi$ is the additive inverse of the complex number $a + bi$.

73. Prove the distributive property for the set of complex numbers.

74. For what values of x is $\sqrt{x - 3}$ a real number?

75. For what values of y is $\sqrt{2y - 10}$ a real number?

76. Perform the multiplications and simplify.

a. $(x + yi)(x - yi)$

b. $(1 - i)^5$

c. $(1 - \sqrt{3})^4$

d. $[x - (2 + 5i)][x - (2 - 5i)]$

For Exercises 77–85, redo Exercises 49–57 using the i key on your graphing calculator. Remember to use parentheses appropriately! (*Note:* The values of a and b in each answer will be in decimal form.)

86. *Mathematics in Writing:* Consider the addition and the multiplication of complex numbers. How does i differ from a variable like x? If you always treat i as though it is a variable, at what step in the procedures of addition or multiplication would you run into trouble?

Chapter Summary

Terms and Symbols

absolute value, $\lvert\ \rvert$	15	factoring	2	polynomial	22
algebraic expression	2	imaginary number	70	power	20
algebraic fraction	40	imaginary part	70	prime polynomial	36
algebraic operations	20	imaginary unit i	69	principal square root	60
base	20	inequalities	13	radical form	60
cancellation principle	41	inequality symbols, $<, >, \leq, \geq$	13	radical sign, $\sqrt{\ }$	60
coefficient	22	integers	4	rational expression	40
complex conjugate	72	irrational numbers	4	rational numbers	4
complex fraction	46	irreducible polynomial	38	rationalizing the denominator	63
complex number	68	LCD	43	rationalizing the numerator	60
constant	23	leading coefficient	23	real number line	2
constant term	23	least common denominator (LCD)	43	real numbers	2
degree of a monomial	23	like terms	23	real part	70
degree of a polynomial	23	member of a set, $\in$	3	scientific notation	2
distance from point A to point B, $\overline{AB}$	17	monomial	22	set	3
element of a set, $\in$	2	natural numbers	2	set notation	3
equal	4	nonnegative numbers	13	simplified form of a radical	59
equivalent fractions	45	not a set member, $\notin$	3	subset	3
evaluate	11	nth root	57	term	2
exponent	20	origin	11	variable	2
factor	1			zero polynomial	23

Key Ideas for Review

Topic	Page	Key Idea
Set	3	A set is a collection of objects or numbers.
The Set of Real Numbers	3	The set of real numbers is composed of the rational and irrational numbers. The rational numbers are those that can be written as the ratio of two integers, $\frac{p}{q}$, with $q \neq 0$; the irrational numbers cannot be written as the ratio of two integers.
Properties	6	The real number system satisfies a number of important properties, including: closure, commutativity, associativity, identities, inverses, distributivity
Equality	6	If two numbers are identical, we say that they are equal.
Properties	7	Equality satisfies these basic properties: reflexive property, symmetric property, transitive property, substitution property
Real Number Line	12	There is a one-to-one correspondence between the set of all real numbers and the set of all points on the real number line. That is, for every point on the line there is a real number, and for every real number there is a point on the line.
Inequalities	13	Algebraic statements using inequality symbols have geometric interpretations using the real number line. For example, $a < b$ says that a lies to the left of b on the real number line.
Operations	13	Inequalities can be operated on in the same manner as statements involving an equal sign with one important exception: when an inequality is multiplied or divided by a negative number, the direction of the inequality is reversed.
Absolute Value	15	Absolute value specifies distance independent of direction. Four important properties of absolute value are: • $\lvert a\rvert \geq 0$ • $\lvert a\rvert = \lvert -a\rvert$ • $\lvert a-b\rvert = \lvert b-a\rvert$ • $\lvert ab\rvert = \lvert a\rvert\,\lvert b\rvert$
Distance	17	The distance between points A and B whose coordinates are a and b, respectively, is given by $\overline{AB} = \lvert b-a\rvert$
Polynomials	22	Algebraic expressions of the form $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ are called polynomials.

Key Ideas for Review

Topic	Page	Key Idea
Operations	23	To add (subtract) polynomials, just add (subtract) like terms. To multiply polynomials, form all possible products, using the rule for exponents: $$a^m a^n = a^{m+n}$$
Factoring	31	A polynomial is said to be factored when it is written as a product of polynomials of lower degree.
Rational Expressions	40	Most of the rules of arithmetic for handling fractions carry over to rational expressions. For example, the LCD has the same meaning except that we deal with polynomials in factored form rather than with integers.
Exponents	49	The rules for positive integer exponents also apply to zero, negative integer exponents, and, in fact, to all rational exponents.
Scientific Notation	52	A number in scientific notation is of the form $$\pm\, a \times 10^m$$ where $1 \leq a < 10$ and m is some integer.
Radicals	60	Radical notation is another way of writing a rational exponent. That is, $$\sqrt[n]{b} = b^{1/n}$$
Principal nth Root	57	If n is even and b is positive, there are two real numbers a such that $b^{1/n} = a$. Under these circumstances, we insist that the nth root be positive. That is, $\sqrt[n]{b}$ is a positive number if n is even and b is positive. Thus $\sqrt{16} = 4$. Similarly, we must write $$\sqrt{x^2} = \lvert x \rvert$$ to ensure that the result is a positive number.
Simplifying	62	To be in simplified form, a radical must satisfy the following conditions: • $\sqrt[n]{x^m}$ has $m < n$. • $\sqrt[n]{x^m}$ has no common factors between m and n. • The denominator has been rationalized.
Complex Numbers	68	Complex numbers were created because there were no real numbers that satisfy a polynomial equation such as $$x^2 + 5 = 0$$

Topic	Page	Key Idea
Imaginary Unit i	69	Using the imaginary unit $i = \sqrt{-1}$, a complex number is of the form $a + bi$, where a and b are real numbers; the real part of $a + bi$ is a and the imaginary part of $a + bi$ is b.
Real Number System	70	The real number system is a subset of the complex number system.

Review Exercises

Solutions to exercises whose numbers are in **blue** are in the Solutions section in the back of the book.

In Exercises 1–3, write each set by listing its elements within braces.

1. The set of natural numbers from -5 to 4, inclusive
2. The set of integers from -3 to -1, inclusive
3. The subset of $x \in S$, $S = \{0.5, 1, 1.5, 2\}$ such that x is an even integer

For Exercises 4–7, determine whether the statement is true (T) or false (F).

4. $\sqrt{7}$ is a real number.
5. -35 is a natural number.
6. -14 is not an integer.
7. 0 is an irrational number.

In Exercises 8–11, identify the property of the real number system that justifies the statement. All variables represent real numbers.

8. $3a + (-3a) = 0$
9. $(3 + 4)x = 3x + 4x$
10. $2x + 2y + z = 2x + z + 2y$
11. $9x \cdot 1 = 9x$

In Exercises 12–14, sketch the given set of numbers on a real number line.

12. The negative real numbers
13. The real numbers x such that $x > 4$
14. The real numbers x such that $-1 \le x < 1$
15. Find the value of $|-3| - |1 - 5|$.
16. Find $\overline{PQ}$ if the coordinates of P and Q are $\frac{9}{2}$ and 6, respectively.
17. A salesperson receives $3.25x + 0.15y$ dollars, where x is the number of hours worked and y is the number of miles driven. Find the amount due the salesperson if $x = 12$ hours and $y = 80$ miles.
18. Which of the following expressions are not polynomials?

    a. $-2xy^2 + x^2y$ b. $3b^2 + 2b - 6$

    c. $x^{-1/2} + 5x^2 - x$ d. $7.5x^2 + 3x - \frac{1}{2}x^0$

In Exercises 19 and 20, indicate the leading coefficient and the degree of each polynomial.

19. $-0.5x^7 + 6x^3 - 5$
20. $2x^2 + 3x^4 - 7x^5$

In Exercises 21–23, perform the indicated operations.

Review Exercises

21. $(3a^2b^2 - a^2b + 2b - a) - (2a^2b^2 + 2a^2b - 2b - a)$

22. $x(2x - 1)(x + 2)$

23. $3x(2x + 1)^2$

In Exercises 24–29, factor each expression.

24. $2x^2 - 2$

25. $x^2 - 25y^2$

26. $2a^2 + 3ab + 6a + 9b$

27. $4x^2 + 19x - 5$

28. $x^8 - 1$

29. $27r^6 + 8s^6$

In Exercises 30–33, perform the indicated operations and simplify.

30. $\dfrac{14(y-1)}{3(x^2-y^2)} \cdot \dfrac{9(x+y)}{-7xy^2}$

31. $\dfrac{4-x^2}{2y^2} \div \dfrac{x-2}{3y}$

32. $\dfrac{x^2-2x-3}{2x^2-x} \div \dfrac{x^2-4x+3}{3x^3-3x^2}$

33. $\dfrac{a+b}{a+2b} \cdot \dfrac{a^2-4b^2}{a^2-b^2}$

In Exercises 34–37, find the LCD.

34. $\dfrac{-1}{2x^2}, \quad \dfrac{2}{x^2-4}, \quad \dfrac{3}{x-2}$

35. $\dfrac{4}{x}, \quad \dfrac{5}{x^2-x}, \quad \dfrac{-3}{(x-1)^2}$

36. $\dfrac{2}{(x-1)y}, \quad \dfrac{-4}{y^2}, \quad \dfrac{x+2}{5(x-1)^2}$

37. $\dfrac{y-1}{x^2(y+1)}, \quad \dfrac{x-2}{2xy-2x}, \quad \dfrac{3x}{4y^2+8y+4}$

In Exercises 38–41, perform the indicated operations and simplify.

38. $2 + \dfrac{4}{a^2-4}$

39. $\dfrac{3}{x^2-16} - \dfrac{2}{x-4}$

40. $\dfrac{\dfrac{3}{x+2} - \dfrac{2}{x-1}}{x-1}$

41. $x^2 + \dfrac{\dfrac{1}{x}+1}{x-\dfrac{1}{x}}$

In Exercises 42–50, simplify and express the answers using only positive exponents. All variables are positive numbers.

42. $(2a^2b^{-3})^{-3}$

43. $2(a^2 - 1)^0$

44. $\left(\dfrac{x^3}{y^{-6}}\right)^{-4/3}$

45. $\dfrac{x^{3+n}}{x^n}$

46. $\sqrt{80}$

47. $\dfrac{2}{\sqrt{12}}$

48. $\sqrt{x^7y^5}$

49. $\sqrt[4]{32x^8y^6}$

50. $\dfrac{\sqrt{x}}{\sqrt{x}+\sqrt{y}}$

51. Compute

$$\frac{(5.10 \times 10^7)(3.45 \times 10^{-2})}{7.10 \times 10^4}$$

to two decimal places and express the answer in scientific notation.

52. Rationalize the numerator for

$$\frac{\sqrt{x}-\sqrt{y}}{x-y}$$

In Exercises 53 and 54, perform the indicated operations. Simplify the answer.

53. $\sqrt[4]{x^2y^2} + 2\sqrt[4]{x^2y^2}$

54. $(\sqrt{3} + \sqrt{5})^2$

55. Evaluate the given expressions using your calculator.

a. $\dfrac{12}{5} - \dfrac{3}{7}$

b. $|(-4)^3 - 5^6|$

c. $\sqrt{8}$

d. π^8

e. $\sqrt[5]{-27}$

f. $\dfrac{|2+\sqrt{3}|}{-6}$

g. $\sqrt[3]{4} + \sqrt{\frac{1}{8}}$ h. $\sqrt[10]{0.5}$

i. $\left(\frac{2}{3}\right)^4$ j. $9^{5/8}$

56. Solve for x and y:

$$(x - 2) + (2y - 1)i = -4 + 7i$$

57. Simplify i^{47}.

In Exercises 58–61, perform the indicated operations and write all answers in the form $a + bi$.

58. $2 + (6 - i)$ 59. $(2 + i)^2$

60. $(4 - 3i)(2 + 3i)$ 61. $\dfrac{4 - 3i}{2 + 3i}$

62. Perform the indicated operations.

a. Combine into one term with a common denominator

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

b. Simplify the quotient

$$\frac{\frac{1}{a} + \frac{1}{b}}{\frac{1}{c} + \frac{1}{d}}$$

63. Dan, at 200 pounds, wishes to reduce his weight to 180 pounds in time to attend his college reunion in 8 weeks. He learns that it takes 2400 calories per day to maintain his weight. A reduction of his caloric intake to 1900 calories per day will result in his losing weight at the rate of 1 pound per week. What should his daily caloric intake be to achieve this goal?

64. The executive committee of the student government association consists of a president, vice-president, secretary, and treasurer.

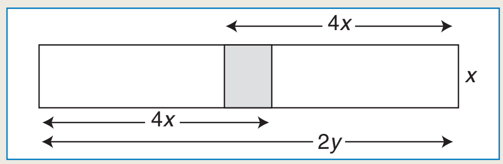
a. In how many ways can a committee of three persons be formed from among the executive committee members?

b. According to the by-laws, there must be at least three affirmative votes to carry a motion. If the president automatically has two votes, list all the minimal winning coalitions.

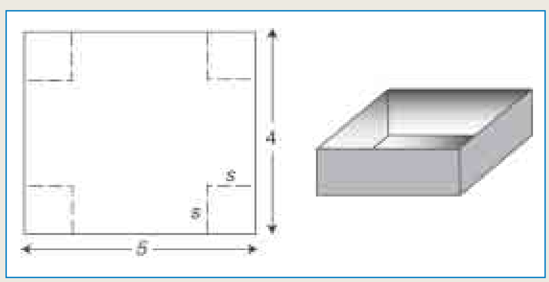
65. If 6 children can devour 6 hot dogs in $\frac{1}{10}$ of an hour, how many children would it take to devour 100 hot dogs in 6000 seconds?

66. A CD player costs a dealer \$80. If he wishes to make a profit of at least 25% of his cost, what must be the lowest selling price for the player?

67. Find the area of the shaded rectangle.



68. An open box is to be made from a 4 feet × 5 feet piece of tin by cutting out squares of equal size from the four corners and bending up the flaps to form sides. Find a formula for the volume in terms of s, the side of the square. Write the inequality that describes the restriction on s.



Review Exercises

69. Compute the following products:

a. $(x - y)(x^2 + xy + y^2)$

b. $(x - y)(x^3 + x^2y + xy^2 + y^3)$

c. $(x - y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4)$

70. Using Exercise 69, find a general formula that allows you to factor $x^n - y^n$, where n is a positive integer.

71. In ancient Alexandria, numbers were multiplied by using an abacus as follows:

$$\begin{aligned} 19 \times 28 &= (20 - 1)(30 - 2) \\ &= (20)(30) - (20)(2) - 30 + 2 \\ &= 600 - 40 - 30 + 2 \\ &= 532 \end{aligned}$$

Set up a comparable sequence of steps for 13×17.

72. Find two ways of grouping and then factoring $ac + ad - bc - bd$.

73. The following calculation represents a sum. If each letter represents a different digit, find the appropriate correspondence between letters and digits so that the sum is correct.

$$\begin{array}{r} \text{FORTY} \\ \text{TEN} \\ \text{TEN} \\ \hline \text{SIXTY} \end{array}$$

74. A natural number is said to be perfect if it is the sum of its divisors other than itself. For example, 6 is the first perfect number since $6 = 1 + 2 + 3$. Show that 28 is the second perfect number.

Every number of the form $2^{p-1}(2^p - 1)$, where $2^p - 1$ is prime, is an even perfect number. (Check your answer when $p = 2$.) Find the third and fourth even perfect numbers. The ancient Greeks could not find the fifth even perfect number. See if you can.

75. The speed of light is 3×10^8 meters per second. Write all answers using scientific notation.

a. How many seconds does it take an object traveling at the speed of light to go 1×10^{26} meters?

b. How many seconds are there in 1 year of 365 days?

c. Write the answer to part (a) in years. (This answer is the approximate age of the universe.)

76. Write $\sqrt{x + \sqrt{x + \sqrt{x}}}$ using exponents.

77. Determine if $(\sqrt{5} - \sqrt{24})^2$ and $(\sqrt{2} - \sqrt{3})^2$ have the same value.

78. The irrational number called the golden ratio

$$T = \frac{\sqrt{5} + 1}{2}$$

has properties that have intrigued artists, philosophers, and mathematicians through the ages. Show that T satisfies the identity

$$T = 1 + \frac{1}{T}$$

79. Rationalize the numerator in the following:

a. $\dfrac{\sqrt{x + h + 1} - \sqrt{x - 1}}{h}$

b. $\dfrac{\sqrt{3 + x} - \sqrt{3}}{x}$

80. In alternating-current theory, the current I (amps), voltage V (volts), and impedance Z (ohms) are treated as complex numbers. The formula relating these quantities is $V = IZ$. If $I = 2 - 3i$ amps and $Z = 6 + 2i$ ohms, find the voltage across this part of the circuit.

Review Tests

In Problems 1 and 2, write each set by listing its elements within braces.

1. The set of positive, even integers less than 13
2. The subset of $x \in S$, $S = \{-1, 2, 3, 5, 7\}$, such that x is a multiple of 3

In Problems 3 and 4, determine whether the statement is true (T) or false (F).

3. -1.36 is an irrational number.
4. π is equal to $\frac{22}{7}$.

In Problems 5 and 6, identify the property of the real number system that justifies the statement. All variables represent real numbers.

5. $xy(z + 1) = (z + 1)xy$
6. $(-6)\left(-\dfrac{1}{6}\right) = 1$

In Problems 7 and 8, sketch the given set of numbers on a real number line.

7. The integers that are greater than -3 and less than or equal to 3
8. The real numbers x such that $-2 \le x < \dfrac{1}{2}$
9. Find the value of $|2 - 3| - |4 - 2|$.
10. Find $\overline{AB}$ if the coordinates of A and B are -6 and -4, respectively.
11. The area of a region is given by the expression $3x^2 - xy$. Find the area if $x = 5$ meters and $y = 10$ meters.
12. Evaluate the expression
$$\frac{-|y-2x|}{|xy|}$$
if $x = 3$ and $y = -1$.

13. Which of the following expressions are not polynomials?

a. x^5 b. $5x^{-4}y + 3x^2 - y$

c. $4x^3 + x$ d. $2x^2 + 3x^0$

In Problems 14 and 15, indicate the leading coefficient and the degree of each polynomial.

14. $-2.2x^5 + 3x^3 - 2x$
15. $14x^6 - 2x + 1$

In Problems 16 and 17, perform the indicated operations.

16. $3xy + 2x + 3y + 2 - (1 - y - x + xy)$
17. $(a + 2)(3a^2 - a + 5)$

In Problems 18 and 19, factor each expression.

18. $8a^3b^5 - 12a^5b^2 + 16a^2b$
19. $4 - 9x^2$

In Problems 20 and 21, perform the indicated operations and simplify.

20. $\dfrac{m^4}{3n^2} \div \left(\dfrac{m^2}{9n} \cdot \dfrac{n}{2m^3}\right)$
21. $\dfrac{16 - x^2}{x^2 - 3x - 4} \cdot \dfrac{x - 1}{x + 4}$
22. Find the LCD of
$$\frac{-1}{2x^2} \qquad \frac{2}{4x^2 - 4} \qquad \frac{3}{x - 2}$$

In Problems 23 and 24, perform the indicated operations and simplify.

23. $\dfrac{2x}{x^2 - 9} + \dfrac{5}{3x + 9}$
24. $\dfrac{2 - \dfrac{4}{x + 1}}{x - 1}$

Review Tests

In Problems 25–28, simplify and express the answers using only positive exponents.

25. $\left(\dfrac{x^{7/2}}{x^{2/3}}\right)^{-6}$

26. $\dfrac{y^{2n}}{y^{n-1}}$

27. $\dfrac{-1}{(x-1)^0}$

28. $(2a^2b^{-1})^2$

In Problems 29–31, perform the indicated operations.

29. $3\sqrt[3]{24} - 2\sqrt[3]{81}$

30. $(\sqrt{7} - 5)^2$

31. $\dfrac{1}{2}\sqrt{\dfrac{xy}{4}} - \sqrt{9xy}$

32. For what values of x is $\sqrt{2 - x}$ a real number?

In Problems 33–35, perform the indicated operations and write all answers in the form $a + bi$.

33. $(2 - i) + (-3 + i)$

34. $(5 + 2i)(2 - 3i)$

35. $\dfrac{5 + 2i}{2 - i}$

Writing Exercises

1. Evaluate $(8)(1.4142)$ and $(8)(\sqrt{2})$. Are these results close to one another? Why?
2. Discuss the need for the complex number system.
3. Compare and contrast the properties of the complex numbers with those of the real numbers.
4. Discuss why division by zero is not permitted.

Chapter 1 Project

Polynomial expressions are used by physicists to study the motion of objects in free fall. Free fall means that the attraction of gravity is the only force operating on the object. In reality, other forces like air resistance play a role.

Take a look at Exercises 86 and 87 in Section 1.3 and Exercises 84–86 in Section 1.4. Set up a table for various planets or moons in our solar system, and use the Internet or other resources to find the data you need to write free-fall equations for objects on those worlds. (*Hint:* The value of a is all you need.) Here are some values to start you off:

$$\text{Mars: } a = 3.72$$

$$\text{Earth: } a = 4.9$$

$$\textit{The Moon}: a = 1.6$$

All these values are in SI units, so the accelerations given above are in meters per second squared.

Try to redo the Exercises listed above for various planets. Write a paragraph explaining the problem described in the chapter opener.