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Existence and Stability of Nontrivial Periodic Solutions of Periodically Forced Discrete Dynamical Systems

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The local existence and local asymptotic stability of nontrivial p-periodic solutions of p-periodically forced discrete systems are proven using Liapunov-Schmidt methods. The periodic solutions bifurcate transcritically from the trivial solution at the critical value $n = n_{cr}$ of the bifurcation parameter with a typical exchange of stability. If the trivial solution loses (gains) stability as n is increased through n_{cr}, then the periodic solutions on the nontrivial bifurcating branch are locally asymptotically stable if and only if they correspond to $n > n_{cr}$ $(n < n_{cr})$.

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1. INTRODUCTION

Let p be a positive integer and x be a sequence $\{x(t)\}_{t=0}^{\infty}$ of vectors in R^m. Define $N \doteq \{0, 1, 2, \ldots\}$. Consider the nonlinear periodically forced discrete system

$$x(t+1) = \bar{F}(n, t, x(t)) \tag{1}$$

where $\bar{F}: R \times N \times R^m \to R^m$, and $\bar{F}(n, t, 0) = 0$ and $\bar{F}(n, t, +p, a) = \bar{F}(n, t, a)$ for all $a \in R^m$, $n \in R$, and $t \in N$.

The purpose of this paper is to study the local existence and stability of solutions of (1) near a critical value $n = n_{cr}$ of the parameter n. Under certain assumptions the trivial solution loses (gains) stability as n is increased through n_{cr}. It will be shown using Liapunov-Schmidt methods [2],[3],[4] that at this critical value a branch of nontrivial p-periodic solutions bifurcates transcritically from the trivial solution. The solutions on this branch are locally asymptotically stable if and only if they correspond to $n > n_{cr}$ $(n < n_{cr})$. Two examples will be given.

The following hypotheses will be used in various parts of the development:

A1) *$\bar{F}$ has the form*

$$\bar{F}(n, t, x(t)) = nF(t, x(t)) + G(t, x(t))$$

where $F, G: N \times R^m \to R^m$, and both F and G are p-periodic in t and vanish at $x =$ 0.

There exist $n_{cr} \in R$ and neighborhoods $U \subseteq R$ of n_{cr} and $V \subseteq R^m$ of zero such that:

A2) *For all $t \in N$, F and G are continuous functions of x on V and are continuously Fréchet differentiable on V with respect to x.*

Define

$$\Phi_{n,x}(t) \doteq \prod_{m=1}^{t} (nF' + G')(t - m, x(t - m)),$$

where prime denotes the Jacobian with respect to x.

A3) *$\Phi_{n_{cr},0}(p)$ has a strictly dominant eigenvalue of one.*

A4) dim *[Ker $(I - \Phi_{n_{cr},0}(p))$] = 1.*

A5) *For all $n \in U$, the p^{th} composite of $\bar{F}$ is a diffeomorphism on V. (It is sufficient that $\Phi_{n_{cr},0}(p)$ have nonzero eigenvalues.)*

A6) $\frac{d}{dn}[l\Phi_{n,0}(p)v_0]_{n=n_{cr}} \neq 0$, *where l and v_0 are left and right eigenvectors of $\Phi_{n_{cr},0}(p)$ belonging to eigenvalue one such that $lv_0 = 1$.*

A7) *For all $t \in N$, F and G are twice continuously Fréchet differentiable on V.*

The first four hypotheses A1–A4 will be used to establish local existence of the bifurcating branch of nontrivial periodic solutions. Hypotheses A5–A7 will be used to verify the exchange of stability between the trivial solution and the bifurcating branch at $n = n_{cr}$. In particular, A6 will guarantee the trivial solution changes stability at $n = n_{cr}$, and the sign of the expression in A6 will indicate whether the trivial solution loses or gains stability. Hypothesis A1 will be relaxed in Section 6.

In order to accomplish these goals, it is necessary to first develop a linear theory.

2. LINEAR THEORY

Let B_p denote the Hilbert space of p-periodic sequences $x = \{x(t)\}_{t=0}^{\infty}$ of vectors in R^m, with inner product

$$\langle x, y \rangle = \sum_{t=0}^{p-1} x(t) \cdot y(t)$$

and norm

$$\| x \| = \sqrt{\langle x, x \rangle}.$$

Let A: $N \to R^m \times R^m$. Consider the homogeneous and nonhomogeneous periodically forced linear equations

$$x(t + 1) = A(t)x(t) \tag{2}$$

and

$$x(t + 1) = A(t)x(t) + h(t) \tag{3}$$

where $h \in B_p$ and $A(t + p) = A(t)$ for all $t \in N$.

Let $\Phi(t)$ be the fundamental solution matrix for (2) defined by

$$\Phi(t+1) = A(t)\Phi(t)$$
$$\Phi(0) = I.$$

Then $\Phi(t) = A(t-1)A(t-2)\cdots A(0)$. The general solution of (2) is $\Phi(t)c$, where $c \in R^m$ is an arbitrary constant.

It is easy to verify that the general solution of (3) is

$$x(t) = \Phi(t)x_0 + \sum_{j=0}^{t-1} \Phi(t, j+1)h(j),$$

where $\Phi(t, j) \doteq A(t-1)A(t-2)\cdots A(j)$. Note that $\Phi(t) = \Phi(t, 0)$.

Equation (2) has a nontrivial solution in B_p if and only if

$$v_0 = \Phi(p)v_0 \tag{4}$$

has a nontrivial solution $v_0 \in R^m$. Equation (3) has a nontrivial solution in B_p if and only if

$$(I - \Phi(p))x_0 = \sum_{j=0}^{p-1} \Phi(p, j+1)h(j) \tag{5}$$

has a nontrivial solution $x_0 \in R^m$.

From this one can conclude:

Theorem 1 *(A Fredholm Alternative) Assume* dim $[Ker(I - \Phi(p))] \lneq 1$. *Either* ☆

a) (2) has no nontrivial solution in B_p and there exists a unique solution of (3) in B_p with $x_0 = (I - \Phi(p))^{-1} \sum_{j=0}^{p-1} \Phi(p, j+1)h(j)$; or

b) (2) has a nontrivial solution in B_p, in which case there exists a solution of (3) in B_p if and only if

$$l \sum_{j=0}^{p-1} \Phi(p, j+1)h(j) = 0, \tag{6}$$

where l is a left eigenvector of $\Phi(p)$ belonging to eigenvalue one [13].

Assume dim $[Ker(I - \Phi(p))] = 1$. Define the linear operator L: $B_p \to B_p$ by

$$Lx \doteq y,$$

where $y \in B_p$ is defined to be the sequence with $y(t) \doteq x(t+1) - A(t)x(t)$.

To see that L is bounded, note that each $A(t)$ for $t = 0, 1, \ldots, p - 1$ is a bounded linear operator on R^m under the Euclidean norm $|\cdot|$. Thus, there exists an $M < \infty$ such that

$$|A(t)x(t)| \leq M\,|x(t)|$$

for all $t = 0, 1, \ldots, p - 1$. Then

$$\begin{aligned}\|\{A(t)x(t)\}_{t=0}^{\infty}\|^2 &= \sum_{t=0}^{p-1} |A(t)x(t)|^2 \\ &\leq M^2 \sum_{t=0}^{p-1} |x(t)|^2 \\ &= M^2\,\|x\|^2,\end{aligned}$$

and so, by the Cauchy-Schwartz Inequality,

$$\begin{aligned}\|Lx\|^2 &= \|\{x(t+1)\} - \{A(t)x(t)\}\|^2 \\ &\leq \|\{x(t+1)\}\|^2 + 2|\langle\{x(t+1)\}, \{A(t)x(t)\}\rangle| + \|\{A(t)x(t)\}\|^2 \\ &\leq \|x\|^2 + 2\|x\|\,M\,\|x\| + M^2\,\|x\|^2 \\ &= (1 + 2M + M^2)\,\|x\|^2.\end{aligned}$$

Since L is bounded, $Ker(L)$ is closed. If a) holds in the Fredholm Alternative, then $Ker(L) = \{0\}$. If b) holds, then $Ker(L) = \{cv \in B_p \mid c \in R\}$, where $v_0 = v(0) \in R^m$ is a right eigenvector of $\Phi(p)$ belonging to eigenvalue one and v is defined by $v(t + 1) \doteq A(t)v(t)$ for all $t \in N$.

Define the linear operator $P_K\colon B_p \to Ker(L)$ by

$$P_K x = \frac{\langle x, v\rangle}{\|v\|^2}\, v.$$

P_K is bounded by the Cauchy-Schwartz Inequality:

$$\|P_K x\| = \frac{|\langle x, v\rangle|}{\|v\|^2}\,\|v\| \leq \frac{\|x\|\,\|v\|}{\|v\|^2}\,\|v\| = \|x\|.$$

P_K is clearly idempotent, and also $Ker(P_K) = (Ran(P_K))^{\perp}$ since $Ker(P_K) = \{x \in B_p \mid \langle x, v\rangle = 0\}$. Thus, P_K is the orthogonal projection onto $Ran(P_K) = Ker(L)$. The operator $I - P_K$ is the orthogonal projection onto the subspace $Ker(P_K) = (Ker(L))^{\perp}$.

Thus B_p is isomorphic to the direct sum $Ker(L) \oplus (Ker(L))^{\perp}$ of orthogonal subspaces, and so each $x \in B_p$ has a unique representation $x = P_K x + (I - P_K)x =$

$$\frac{\langle x, v \rangle}{\|v\|^2} v + (x - \frac{\langle x, v \rangle}{\|v\|^2} v) \tag{7}$$

[1].

If $Lx = Ly$, then $x - y \in Ker(L)$. Thus, when L is restricted to $(Ker(L))^{\perp}$, it is a bijection onto $Ran(L)$ and has a right inverse L^{-1}: $Ran(L) \to (Ker(L))^{\perp}$.

3. NONLINEAR THEORY

Given hypotheses A1 and A2, Equation (1) can be expanded about zero as

$$x(t+1) = [nF'(t, 0) + G'(t, 0)]\, x(t) + nH(t, x(t)) + K(t, x(t)) \tag{8}$$

where the prime denotes the Jacobian and H and K are $O(\|x\|^2)$ near $\|x\| = 0$. Note that $\Phi_{n_{cr},0}$ is the fundamental solution matrix for the homogeneous linear equation

$$x(t+1) = [n_{cr} F'(t, 0) + G'(t, 0)]\, x(t). \tag{9}$$

Let S denote the set of sequences of elements of R^m. Define the operator B: $R \times B_p \to S$ by

$$B(n, x) = (n - n_{cr})\{F'(t, 0)x(t)\}_{t=0}^{\infty} + n\{H(t, x(t))\}_{t=0}^{\infty} + \{K(t, x(t))\}_{t=0}^{\infty}.$$

Equations (1) and (8) are equivalent (in B_p) to the operator equation

$$Lx = B(n, x), \tag{10}$$

where $Lx = \{x(t+1) - [n_{cr}F'(t, 0) + G'(t, 0)]\, x(t)\}_{t=0}^{\infty}$.

Under assumption A3, $\Phi_{n_{cr},0}(p)$ has a strictly dominant eigenvalue of one; hence the equation $Lx = 0$ has nontrivial solutions in B_p. Given assumption A4, $Ker(L) = \{cv \in B_p \mid c \in R\}$, where $v_0 = v(0)$ is a right eigenvector of $\Phi_{n_{cr},0}(p)$ belonging to eigenvalue one and $v(t+1) \doteq (n_{cr}F' + G')\,(t, 0)v(t)$ for all $t \in N$.

Note that by Theorem 1, $B(n, x) \in Ran(L)$ if and only if there exists $y \in B_p$ such that y solves

$$\begin{aligned} y(t+1) = {} & (n_{cr}F' + G')\,(t, 0)y(t) \\ & + (n - n_{cr})F'(t, 0)x(t) + nH(t, x(t)) + K(t, x(t)) \end{aligned}$$

if and only if the orthogonality condition

$$l \sum_{j=0}^{p-1} \Phi_{n_{cr},0}(p, j+1)\, [(n - n_{cr})F'(j,0)x(j) + nH(j, x(j)) + K(j, x(j))] = 0 \qquad (11)$$

holds, where l is a left eigenvector of $\Phi_{n_{cr},0}(p)$ belonging to eigenvalue one.

4. BIFURCATION OF P-PERIODIC SOLUTIONS AT $n = n_{cr}$

Theorem 2 *Suppose Equation (1) satisfies hypotheses A1-A4. Then there exists a unique branch of nontrivial solutions $(n, x) \in R \times B_p$ of the equation*

$$Lx = B(n, x)$$

in a sufficiently small neighborhood of $(n_{cr}, 0)$.

Proof Assume that solutions $(n, x) \in R \times B_p$ of Equation (10) which are near the $(n_{cr}, 0)$ solution have the form

$$x = \epsilon v + \epsilon w(\epsilon)$$

$$n = n_{cr} + \epsilon\lambda(\epsilon),$$

where $\epsilon \in R$, $\epsilon \approx 0$, $w(\epsilon) \in (Ker(L))^{\perp}$, and $w(0) = 0$. Substitution of these expansions into Equation (10) yields equations for first order and higher order terms:

$$\epsilon L(v) = 0$$
$$\epsilon L(w(\epsilon)) = \epsilon T(\lambda(\epsilon), \epsilon, w(\epsilon))$$

with $\epsilon T(\lambda(\epsilon), \epsilon, w(\epsilon) \doteq B(n_{cr} + \epsilon\lambda(\epsilon), \epsilon v + \epsilon w(\epsilon))$. Note that T is $O(\epsilon)$.

The first equation is satisfied by the definition of v and we can now turn our attention to finding nontrivial solutions of

$$L(w(\epsilon)) = T(\lambda(\epsilon), \epsilon, w(\epsilon)). \qquad (12)$$

Since $Ran(T) \subseteq Ran(L)$ if and only if Equation (11) is satisfied,

$$l \sum_{j=0}^{p-1} \Phi_{n_{cr},0}(p, j+1)\, [\epsilon\lambda(\epsilon)F'(j,0)(\epsilon v_j + \epsilon w_j(\epsilon))] +$$

$$l \sum_{j=0}^{p-1} \Phi_{n_{cr},0}(p, j+1)\, [(n_{cr} + \epsilon\lambda(\epsilon))\, H(j, \epsilon v_j + \epsilon w_j(\epsilon)) + K(j, \epsilon v_j + \epsilon w_j(\epsilon))] = 0,$$

where $v_j \doteq v(j)$, etc. This equation can be solved explicitly for $\lambda = \lambda(\epsilon, w(\epsilon))$ from

$$\lambda(\epsilon)\frac{1}{\epsilon}l\sum_{j=0}^{p-1}\Phi_{n_{cr},0}(p, j+1)\,[F'(j,0)(\epsilon v_j + \epsilon w_j(\epsilon)) + H(j, \epsilon v_j + \epsilon w_j(\epsilon))] =$$

$$-\frac{1}{\epsilon^2}l\sum_{j=0}^{p-1}\Phi_{n_{cr},0}(p, j+1)\,[n_{cr}H(j, \epsilon v_j + \epsilon w_j(\epsilon)) + K(j, \epsilon v_j + \epsilon w_j(\epsilon))], \tag{13}$$

and the result can be substituted back into Equation (12).

It is therefore sufficient to solve

$$w(\epsilon) - L^{-1}T(\lambda(\epsilon, w(\epsilon)), \epsilon, w(\epsilon)) = 0$$

for $w(\epsilon)$.

Let $\Omega(\epsilon, w) \doteq w - L^{-1}T(\lambda(\epsilon, w), \epsilon, w)$. Then $\Omega(0, 0) = 0$, and the Fréchet derivative of Ω with respect to w at $(0, 0)$ is the identity since T is $O(\epsilon)$. By the Implicit Function Theorem, there exists a unique $w = w(\epsilon)$ for sufficiently small $|\epsilon|$ such that

$$\Omega(\epsilon, w(\epsilon)) = 0.$$

5. STABILITY

Given assumption A2, the stability of a p-periodic solution x of

$$x(t+1) = \bar{F}(n, t, x(t)) \tag{14}$$

is equivalent to the stability of the equilibrium solution $\{x_0\}$ of the pth composite map

$$z(t+1) = \bar{F}\big(n, p-1, \bar{F}(n, p-2, \ldots \bar{F}(n, 0, z(t))) \ldots\big). \tag{15}$$

It is therefore sufficient, given assumption A5, to do a linearized stability analysis of Equation (15) at the equilibrium $\{x_0\}$ if $\{x_0\}$ is a hyperbolic equilibrium [12].

Suppose $x \in B_p$ solves Equation (14). Let $z = \{x_0\} + y$, for $y \in B_p$. Then

$$x_0 + y(t+1) = \bar{F}\big(n, p-1, \bar{F}(n, p-2, \ldots \bar{F}(n, 0, x_0)) \ldots\big) + \Phi_{n,x}(p)y(t) + O(\|y\|^2)$$

and hence

$$y(t+1) = \Phi_{n,x}(p)y(t) + O(\|y\|^2).$$

5.1 Stability of the Trivial Solution

In the case of the trivial solution $x = 0$, $\Phi_{n,0}(p)$ has a strictly dominant eigenvalue of one when $n = n_{cr}$ by assumption A3. In fact, assumption A6 ensures the trivial solution changes stability at $n = n_{cr}$, as the next theorem demonstrates.

Theorem 3 *Assume hypotheses A1-A6. Then*

$$\frac{d}{dn}[l\Phi_{n,0}(p)v_0]_{n=n_{cr}} > 0\ (<0)$$

if and only if the trivial solution of (1) loses (gains) stability as n increases through n_{cr}.

Proof By hypothesis A3, $\Phi_{n_{cr},0}(p)$ has a strictly dominant eigenvalue of one. Let l and v_0 be left and right eigenvectors of $\Phi_{n_{cr},0}(p)$ belonging to eigenvalue one such that $lv_0 = 1$.

Since $\Phi_{n,0}(p)$ is continuous in n, $\Phi_{n,0}(p)$ has a strictly dominant eigenvalue for n sufficiently close to n_{cr}. Let γ and q be the strictly dominant eigenvalue and right eigenvector of $\Phi_{n,0}(p)$ so that

$$\Phi_{n,0}(p)q = \gamma q. \tag{16}$$

Expand $\Phi_{n,0}(p)$, q, and γ about $n = n_{cr}$:

$$\Phi_{n,0}(p) = \Phi_{n_{cr},0}(p) + \frac{d\Phi_{n,0}(p)}{dn}\Big|_{n=n_{cr}}(n - n_{cr}) + O\,[|n - n_{cr}|^2]$$
$$q = v_0 + q_1(n - n_{cr}) + O\,[|n - n_{cr}|^2]$$
$$\gamma = 1 + \gamma_1(n - n_{cr}) + O\,[|n - n_{cr}|^2].$$

When the zeroth and first order terms of (16) are equated, we have

$$\Phi_{n_{cr},0}(p)v_0 = v_0$$

$$\Phi_{n_{cr},0}(p)q_1 + \frac{d\Phi_{n,0}(p)}{dn}\Big|_{n=n_{cr}} v_0 = q_1 + \gamma_1 v_0.$$

The first of these equations is satisfied by the definition of v_0, and the second can be written as

$$(\Phi_{n_{cr},0}(p) - I)\,q_1 = \gamma_1 v_0 - \frac{d\Phi_{n,0}(p)}{dn}\Big|_{n=n_{cr}} v_0.$$

Thus,

$$\gamma_1 l v_0 = l\,\frac{d\Phi_{n,0}(p)}{dn}\Big|_{n=n_{cr}} v_0,$$

or

$$\gamma_1 = \frac{d}{dn}\,[l\Phi_{n,0}(p)v_0]\,|_{n=n_{cr}}. \;\square$$

Note that

$$\frac{d}{dn}\,[l\Phi_{n,0}(p)v_0]\,|_{n=n_{cr}} = l\sum_{j=0}^{p-1} \Phi_{n_{cr},0}(p, j+1)F'(j,0)v(j).$$

5.2 Stability of Nontrivial Solutions

Since $\Phi_{n,x}(p)$ is continuous in both n and x for x sufficiently near zero, it has a strictly dominant eigenvalue for $\|x\|$ sufficiently small and n sufficiently close to n_{cr}. For nontrivial solutions $x \in B_p$, let η be the strictly dominant eigenvalue of $\Phi_{n,x}(p)$ and q be a right eigenvector belonging to η. Let

$$\begin{aligned}
x &= \epsilon v + O(\epsilon^2)\\
n &= n_{cr} + \epsilon n_1 + \epsilon^2 r(\epsilon)\\
\eta &= 1 + \epsilon\eta_1 + O(\epsilon^2)\\
q &= v_0 + \epsilon q_1 + O(\epsilon^2)\\
\Phi_{n(\epsilon),\,x(\epsilon)}(p) &= \Phi_{n_{cr},0}(p) + \epsilon\Psi + O(\epsilon^2)
\end{aligned}$$

where v and $\lambda(\epsilon) = n_1 + \epsilon r(\epsilon)$ are as in Theorem 2.

The following Theorem verifies the exchange of stability between the trivial solution and the bifurcating branch of nontrivial periodic solutions under the assumption that $n_1 \neq 0$.

Theorem 4 *Assume hypotheses A1-A7. Then*

$$n_1 = \frac{-\frac{1}{2}\, l\sum_{j=0}^{p-1} \Phi_{n_{cr},0}(p, j+1)\,[(n_{cr}F''(j,0) + G''(j,0))\,(v(j))]\,v(j)}{\frac{d}{dn}\,[l\Phi_{n,0}(p)v_0]_{n=n_{cr}}}, \tag{17}$$

and

$$\eta_1 = -n_1\frac{d}{dn}\,[l\Phi_{n,0}(p)v_0]_{n=n_{cr}},$$

where "$''$" denotes the second Fréchet derivative.

Assume $n_1 \neq 0$. If the trivial solution loses stability as n is increased through $n = n_{cr}$, then for n sufficiently close to n_{cr}, the nontrivial p-periodic solutions (n, x) on the *bifurcating branch in Theorem 2 are locally asymptotically stable if $n > n_{cr}$ and unstable if $n < n_{cr}$.*

If the trivial solution gains stability as n is increased through $n = n_{cr}$, then for n sufficiently close to n_{cr}, the nontrivial p-periodic solutions (n, x) are locally asymptotically stable if $n < n_{cr}$ and unstable if $n > n_{cr}$.

Proof Since $\lambda(0) = n_1$,

$$n_1 l \sum_{j=0}^{p-1} \Phi_{n_{cr},0}(p, j+1)F'(j,0)v(j) \tag{18}$$

$$= -\frac{1}{2} l \sum_{j=0}^{p-1} \Phi_{n_{cr},0}(p, j+1)\,[(n_{cr}F''(j,0) + G''(j,0))\,(v(j))]\,v(j),$$

by Equation (13), and Equation (17) follows.

The rest of the proof will consist of deriving the identity $\eta_1 = -\,n_1 \frac{d}{dn}[l\Phi_{n,0}v_0]_{n=n_{cr}}$ from the eigenvalue equation

$$\Phi_{n,x}(p)q = \eta q \tag{19}$$

by equating first order terms.

The equations for the zeroth and first order terms of (19) are

$$\Phi_{n_{cr},0}(p)v_0 = v_0$$

$$\Psi v_0 + \Phi_{n_{cr},0}(p)q_1 = \eta_1 v_0 + q_1.$$

The first of these is satisfied, and so it is sufficient to consider the equation

$$(\Phi_{n_{cr},0}(p) - I)q_1 = \eta_1 v_0 - \Psi v_0,$$

and hence

$$\eta_1 = l\Psi v_0. \tag{20}$$

It remains to compute $l\Psi v_0$ as a function of n_1. Note that

$$\frac{dx(t)}{d\epsilon}\Big|_{\epsilon=0} = v(t).$$

Thus,

$$l\Psi v_0 = l\frac{d}{d\epsilon}\left(\Phi_{n(\epsilon),x(\epsilon)}(p)\right)_{\epsilon=0} v_0$$

$$= l\frac{d}{d\epsilon}\left(\prod_{m=1}^{p}(nF' + G')(p - m, x(p - m))\right)_{\epsilon=0} v_0$$

$$= l\sum_{j=0}^{p-1}\Phi_{n_{cr},0}(p, j + 1)\,[n_1F'(j, 0) + (n_{cr}F'' + G'')\,(j, 0)\,(v(j))]\,v(j)$$

$$= n_1 l\sum_{j=0}^{p-1}\Phi_{n_{cr},0}(p, j + 1)F'(j, 0)v(j)$$

$$+ l\sum_{j=0}^{p-1}\Phi_{n_{cr},0}(p, j + 1)\,[(n_{cr}F'' + G'')\,(j, 0)\,(v(j))]\,v(j).$$

By Equation (18),

$$l\Psi v_0 = -n_1 l\sum_{j=0}^{p-1}\Phi_{n_{cr},0}(p, j + 1)F'(j, 0)v(j)$$

$$= -n_1\frac{d}{dn}[l\Phi_{n,0}(p)v_0]_{n=n_{cr}}.$$

6. FURTHER GENERALIZATIONS

Now assume $\bar{F}$ in Equation (1) has the more general form

B1)

$$\bar{F}(n, t, x(t)) = (nF(t) + G(t))x(t) + H(n, t, x(t))$$

where F, G: $N \to R^m \times R^m$, $nF + G$ is the Jacobian of $\bar{F}$ at $x = 0$, H: $R \times N \times R^m \to R^m$, and H is $O(\|x\|^2)$.

Under suitable hypotheses analogous to A2-A7, all the theorems in the previous sections hold. The modified hypotheses and some of the modified equations are indicated below.

There exist $n_{cr} \in R$ and neighborhoods U of n_{cr} and V of zero such that:

B2) *For all $x \in V$, $n \in U$, and $t \in N$, $\bar{F}$ is a continuous function of x and n and has a Fréchet derivative which is also continuous in x and n.*

Define

$$\Phi_{n,x}(t) \doteq \prod_{m=1}^{t}\bar{F}'(n, t - m, x(t - m)).$$

B3) $\Phi_{n_{cr},0}(p) = \prod_{m=1}^{p}(n_{cr}F + G)(p - m)$ *has a strictly dominant eigenvalue of one.*

B4) dim $[Ker(I - \Phi_{n_{cr},0}(p))] = 1$.
B5) *For all $n \in U$, the p^{th} composite of $\bar{F}$ is a differomorphism on V.*
B6) $\dfrac{d}{dn}[l\Phi_{n,0}(p)v_0]_{n=n_{cr}} \neq 0$, *where l and v_0 are left and right eigenvectors of $\Phi_{n_{cr},0}(p)$ belonging to eigenvalue one such that $lv_0 = 1$.*
B7) *For all $x \in V$, $n \in U$, and $t \in N$, the second Fréchet derivative of $\bar{F}$ exists and is continuous in x and n.*

In this case, the operators B and L become

$$B(n, x) = (n - n_{cr})\,\{F(t)x(t)\}_{t=0}^{\infty} + \{H(n, t, x(t))\}_{t=0}^{\infty}$$
$$Lx = \{x(t+1) - [n_{cr}F(t) + G(t)]\,x(t)\}_{t=0}^{\infty},$$

and Equation (13) becomes

$$\lambda(\epsilon)\frac{1}{\epsilon}\,l\sum_{j=0}^{p-1}\Phi_{n_{cr},0}(p, j+1)F(j)(\epsilon v_j + \epsilon w_j(\epsilon)) +$$
$$\frac{1}{\epsilon^2}\,l\sum_{j=0}^{p-1}\Phi_{n_{cr},0}(p, j+1)H(n_{cr} + \epsilon\lambda(\epsilon), j, \epsilon v_j + \epsilon w_j(\epsilon)) = 0. \tag{21}$$

This equation cannot be explicitly solved for λ; however, the Implicit Function Theorem can be applied to guarantee the existence of a unique $\lambda = \lambda(\epsilon, \mathrm{w}(\epsilon))$ for sufficiently small $|\epsilon|$.
In Theorem 4, Equation (17) becomes

$$n_1 = \frac{-\frac{1}{2}\,l\sum_{j=0}^{p-1}\Phi_{n_{cr},0}(p, j+1)\,[\bar{F}''(n_{cr}, j, 0)\,(v(j))]\,v(j)}{\frac{d}{dn}[l\Phi_{n,0}(p)v_0]_{n=n_{cr}}},$$

and $l\Psi v_0$ becomes

$$l\Psi v_0 = l\frac{d}{d\epsilon}(\Phi_{n,x}(p))_{\epsilon=0}v_0$$
$$= l\frac{d}{d\epsilon}\left(\prod_{m=1}^{p}(\bar{F}'(n, p-m, x(p-m)))\right)_{\epsilon=0} v_0$$
$$= l\sum_{j=0}^{p-1}\Phi_{n_{cr},0}(p, j+1)\left[\bar{F}''(n_{cr}, j, 0)\,(v(j)) + n_1\frac{d}{dn}\bar{F}(n, j, 0)\,\Big|_{n=n_{cr}}\right]v(j)$$
$$= n_1 l\sum_{j=0}^{p-1}\Phi_{n_{cr},0}(p, j+1)F(j)v(j) + l\sum_{j=0}^{p-1}\Phi_{n_{cr},0}(p, j+1)\,[\bar{F}''(n_{cr}, j, 0)\,(v(j))]\,v(j)$$

$$= -n_1 l \sum_{j=0}^{p-1} \Phi_{n_{cr},0}(p, j+1)F(j)v(j)$$

$$= -n_1 \frac{d}{dn} [l\Phi_{n,0}(p)v_0]_{n=n_{cr}}.$$

7. EXAMPLES

The following examples illustrate bifurcating periodic solutions. That the stability result is local about $n = n_{cr}$ as well as local in the usual dynamical sense is seen in the figures; destabilizing bifurcations can occur as n increases.

7.1 Example 1

The first example is the one-dimensional "discrete logistic" equation with periodic forcing:

$$x(t+1) = n(\cos \pi t + \sqrt{2})x(t)(1 - x(t)).$$

$\cos \pi t + \sqrt{2}$ is periodic in time with period $p = 2$. In this case $n_{cr} = 1$ and

$$\Phi_{1,0}(2) = 1$$

$$\frac{d}{dn} [\Phi_{n,0}(2)]_{n=1} = 2$$

$$n_1 = \frac{\sqrt{2}}{2}$$

$$\eta_1 = -\sqrt{2}.$$

As n is increased through one, the trivial solution loses stability and a positive branch of locally asymptotically stable 2-periodic solutions bifurcates supercritically (see Fig. 1).

7.2 Example 2

The next example comes from population biology. Consider an extended Ebenman model for juvenile and adult competition [5], [9], [10], [11]:

$$x(t+1) = \mu_N m e^{-d\gamma x(t)} e^{-dy(t)} y(t)$$

$$y(t+1) = \mu_J e^{-cx(t)} e^{-c\delta y(t)} x(t) + \mu_A y(t)$$

$$m > 0$$

$$\mu_N, \mu_J, \mu_A \in (0, 1).$$

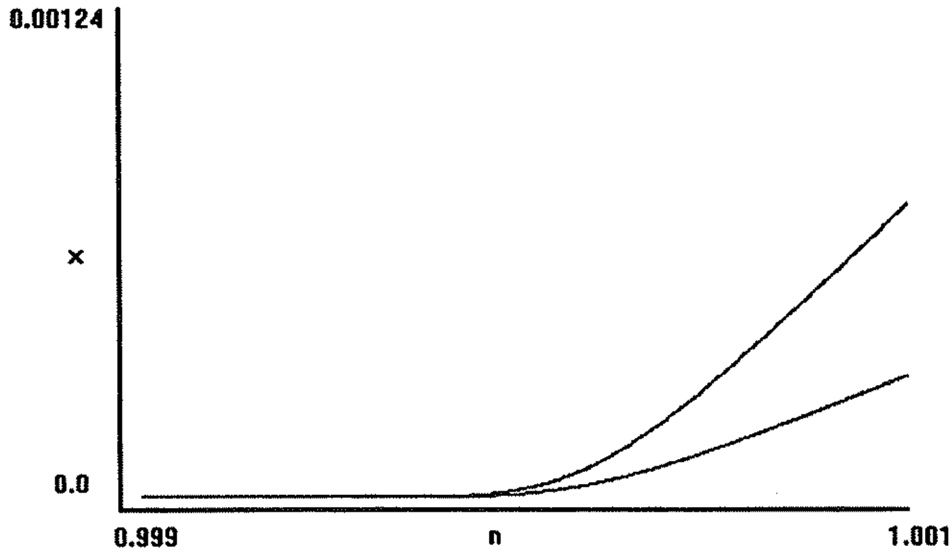

FIGURE 1a. Bifurcation diagram for the periodically forced discrete logistic equation $x(t+1) = n(\cos \pi t + \sqrt{2})x(t)\,(1 - x(t))$; $n = 0.999$ to $n = 1.001$. The trivial solution loses stability at $n_{cr} = 1$ and bifurcates supercritically into a locally asymptotically stable positive 2-cycle.

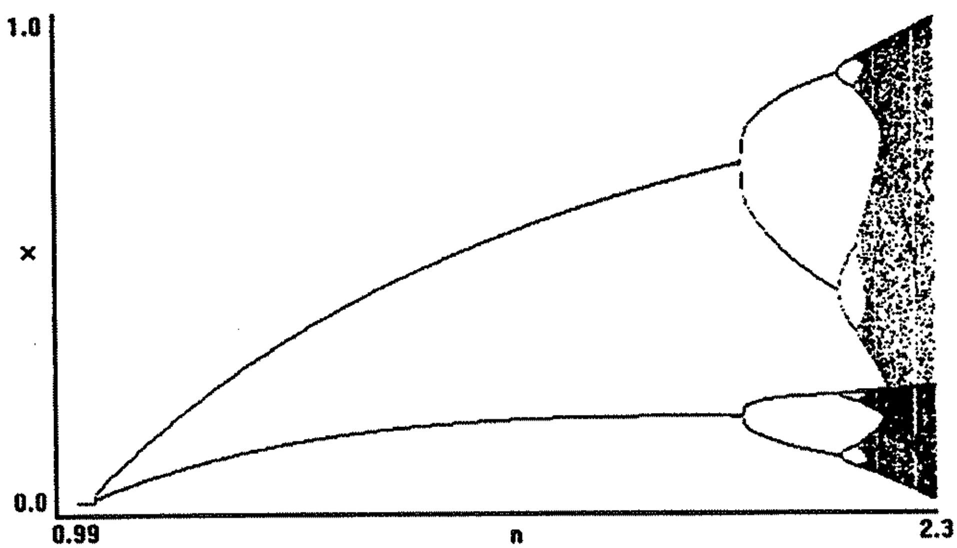

FIGURE 1b. Bifurcation diagram for the periodically forced discrete logistic equation; $n = 0.99$ to $n = 2.3$.

$x(t)$ and $y(t)$ denote the numbers of juveniles and adults, respectively, m is the expected number of offspring born to an adult in one unit of time, and μ_N, μ_J, μ_A are the probabilities that an individual newborn, juvenile, and adult, respectively, will survive one unit of time.

Suppose the birth rate is seasonal with $m(t) = n\,[1 + \alpha(-1)^t]$, for $\alpha \in [0, 1)$. The model can be written in a form satisfying hypothesis A1, and

$$\Phi_{n,0}(2) = \begin{pmatrix} \mu_J\mu_N n(1-\alpha) & \mu_A\mu_N n(1-\alpha) \\ \mu_A\mu_J & \mu_N\mu_J n(1+\alpha) + \mu_A^2 \end{pmatrix}.$$

The eigenvalues are

$$\lambda = \mu_J\mu_N n + \frac{1}{2}\mu_A^2 \pm \frac{1}{2}\sqrt{(\mu_A^2 + 2\mu_N\mu_J n)^2 - 4\mu_J^2\mu_N^2 n^2(1-\alpha^2)}$$

$$= \mu_J\mu_N n + \frac{1}{2}\mu_A^2 \pm \frac{1}{2}\sqrt{(\mu_A^2 + 2\mu_N\mu_J n\alpha)^2 + 4\mu_A^2\mu_N\mu_J n(1-\alpha)}.$$

From these equations one can see that there are two distinct, positive, real eigenvalues, and that there exists a unique $n_{cr} > 0$ for which the dominant eigenvalue equals one. From the characteristic equation it can be shown that

$$n_{cr} = \frac{1 - \sqrt{1 - (1-\alpha^2)(1-\mu_A^2)}}{\mu_J\mu_N(1-\alpha^2)}. \tag{22}$$

Also,

$$v_0 = \begin{pmatrix} \mu_A\mu_N n_{cr}(1-\alpha) \\ 1 - \mu_J\mu_N n_{cr}(1-\alpha) \end{pmatrix}$$

$$l = \frac{1}{\mu_A^2\mu_N\mu_J n_{cr}(1-\alpha) + (1 - \mu_J\mu_N n_{cr}(1-\alpha))^2}\left(\mu_A\mu_J,\ 1 - \mu_J\mu_N n_{cr}(1-\alpha)\right)$$

and

$$\frac{d}{dn}[l\Phi_{n,0}(2)v_0]_{n=n_{cr}} = \frac{\mu_A^2\mu_J\mu_N(1-\alpha) + \mu_J\mu_N(1+\alpha)(1 - \mu_J\mu_N n_{cr}(1-\alpha))^2}{\mu_A^2\mu_J\mu_N(1-\alpha)n_{cr} + (1 - \mu_J\mu_N n_{cr}(1-\alpha))^2} > 0.$$

The formulas for n_1 and η_1 can be calculated but are somewhat unwieldly. From Equation (22) one can see that $1 - \mu_J\mu_N n_{cr}(1-\alpha) > 0$, and hence all the entries of v_0 and l are positive. Since the second Fréchet derivative of $\bar{F}$ is nonpositive, one can see that all the terms in the formula for n_1 are nonnegative. A few calculations show n_1 is, in fact, positive.

Thus, as n is increased through n_{cr}, the trivial solution loses stability and a positive branch of locally asymptotically stable 2-periodic solutions bifurcates supercritically (see Fig. 2).

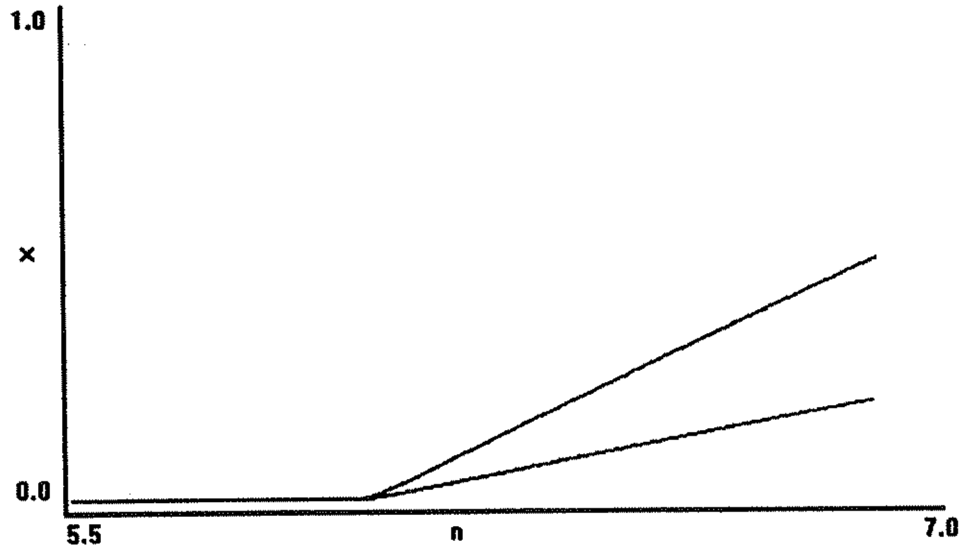

FIGURE 2a. Bifurcation diagram for juveniles in the extended Ebenman model with $\mu_J = 0.2$, $\mu_N = 0.4$, $\mu_A = 0.5$, $\alpha = 0.3$, $c = 0.2$, $d = 0.2$, $\delta = 0.4$, and $\gamma = 0.3$; $n = 5.5$ to $n = 7.0$. The trivial solution loses stability at $n_{cr} \approx 5.99$ and bifurcates supercritically into a locally asymptotically stable positive 2-cycle.

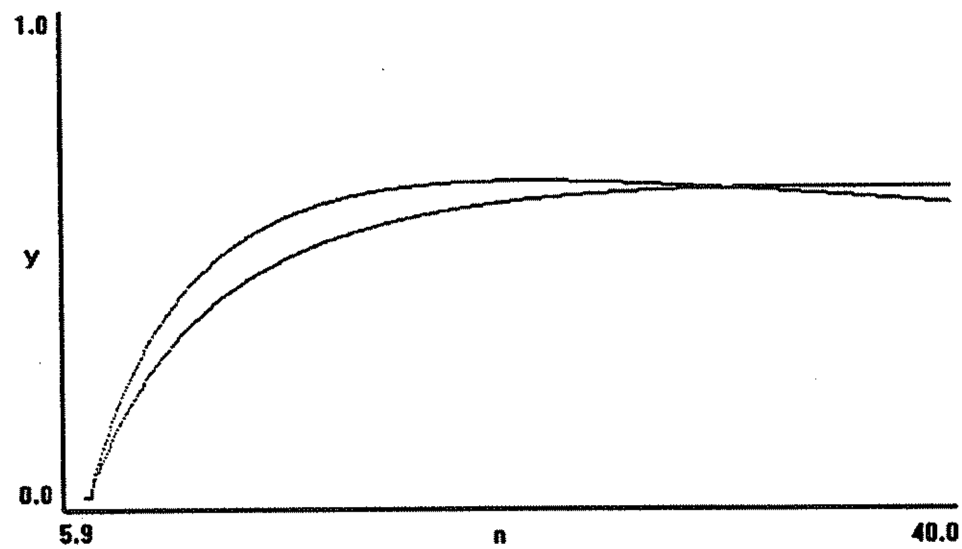

FIGURE 2b. Bifurcation diagram for adults in the extended Ebenman model with $\mu_J = 0.2$, $\mu_N = 0.4$, $\mu_A = 0.5$, $\alpha = 0.3$, $c = 0.2$, $d = 0.2$, $\delta = 0.4$, and $\gamma = 0.3$; $n = 5.9$ to $n = 40.0$.

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