DEFINITION: \( h : X \rightarrow Y \) is a homeomorphism if it is one-one, onto, and bi-continuous.

\[ f : X \rightarrow X, \quad g : Y \rightarrow Y \]

Let \( g \circ h : X \rightarrow Y \) denote the composite of \( h \) and \( g \).

DEFINITION: \( f \) is (topologically) conjugate to \( g \) if there exists a homeomorphism \( h : X \rightarrow Y \) such that \( h \circ f = g \circ h \).

Some facts about conjugate maps:

Assume \( f : X \rightarrow X, \ g : Y \rightarrow Y \) are \( h \)-conjugate

- \( f^{(k)} \) and \( g^{(k)} \) are \( h \)-conjugate
- \( p = \) periodic point of \( g \) \( \Rightarrow \) \( h^{-1}(p) = \) periodic point of \( f \)
- \( D \) dense in \( X \) \( \Rightarrow \) \( h(D) \) dense in \( Y \)
- \( g \) transitive \( \Rightarrow \) \( f \) transitive
- \( g \) chaotic \( \Rightarrow \) \( f \) chaotic

\[ f(x) = h^{-1}(g(h(x))) \quad \forall \ x \in X \]

\[ h(f(x)) = g(h(x)) \quad \forall \ x \in X \]

i.e., \( h \circ f = g \circ h \)
Case 1: the quadratic map when $b = 4$

$x(t+1) = 4x(t)(1-x(t))$

Claim: The map $f(x) = 4x(1-x)$
is conjugate to the tent map $T(x)$

**THEOREM:** $f(x) = 4x(1-x)$ is chaotic on $[0,1]$

It follows that the map has
✓ a dense set of periodic orbits
✓ an orbit dense in the unit interval
✓ sensitivity to initial conditions

Define $h(x) = \sin^2(\pi x/2)$

$h : [0,1] \to [0,1]$ is a homeomorphism
(this follows from $h'(x) > 0$ on $(0,1)$)

$(h \circ T)(x) = \sin^2(\pi T(x)/2) = \begin{cases} \sin^2 \pi x, & 0 \leq x \leq 1/2 \\ \sin^2 \pi (1-x), & 1/2 \leq x \leq 1 \end{cases}$

$= \sin^2 \pi x$

$(f \circ h)(x) = 4\sin^2(\pi x/2)(1-\sin^2(\pi x/2))$

$= (2\sin(\pi x/2)\cos(\pi x/2))^2$

$= (\sin \pi x)^2$

Case 2: the quadratic map when $b > 4$

$x(t+1) = bx(t)(1-x(t))$

$A_n = \bigcup_{s=0}^{s=1} I_{4^{s-1}}$

The points that stay in $[0,1]$ for $n$ steps

$\Lambda = \bigcap_{n=1}^{\infty} A_n$

The points that stay in $[0,1]$ for all $n$

**Facts about $\Lambda$**

1. $\Lambda \neq \emptyset$ (contains 0,1 and endpoints of all $I$’s)
2. $\Lambda$ is bounded (contained in $[0,1]$)
3. $\Lambda$ is closed (intersection of closed intervals)
4. $\Lambda$ contains no intervals
5. $x \in \Lambda \Rightarrow \exists y_n \in \Lambda, y_n \neq x, \text{ such that } \lim_{n \to \infty} y_n = x$
(4) follows from ...

Let \( \alpha \) = fraction removed each step \((0 < \alpha < 1)\)

The \( 2^n \) subintervals at step \( n \) each have length \( \left( \frac{1-\alpha}{2} \right)^n \)

Total length at step \( n \) = \((1-\alpha)^n \to 0 \) as \( n \to \infty \)

(5) follows (4)

Some jargon ...

(2) + (3) = "compact"

(4) = "totally disconnected"

(2) + (5) = "perfect"

DEFINITION: A set \( C \) in \( \mathbb{R}^m \) is a Cantor set if it is compact, totally disconnected and perfect.

Thus, we have shown

THEOREM: \( \Lambda \) is a Cantor set

DEFINITION: A set \( C \) in \( \mathbb{R}^m \) is forward invariant under \( f \) if \( f(C) \) is contained in \( C \). It is invariant if \( f(C) = C \)

THEOREM: When \( b > 4 \) the quadratic map has an invariant Cantor set \( \Lambda \).

What are the dynamics on \( \Lambda \)?

THEOREM: \( f(x) = bx(1-x) \) is chaotic on \( \Lambda \) for \( b > 4 \)

Proof: Difficult for \( 4 < b \leq 2 + \sqrt{5} \). For \( b > 2 + \sqrt{5} \)

define \( h: \Lambda \to \Sigma^1 \) by

\[
h(x) = (a_0, a_1, a_2, \ldots) \quad \text{where} \quad a_n = \begin{cases} 0 & \text{if } x(n) \in I_0 \\ 1 & \text{if } x(n) \in I_1 \end{cases}
\]

the "itinerary" of \( x \)

(1) \( f \) is \( h \)-conjugate to the shift map \( \sigma \)

\[
(h \circ f)(x) = h(f(x)) = (a_0, a_2, a_3, \ldots)
\]

\[
(\sigma \circ h)(x) = \sigma(h(x)) = (a_1, a_2, a_3, \ldots)
\]

(2) \( h \) is a homeomorphism

Uses \( b > 2 + \sqrt{5} \)

Case 3: \( 3.57 < b < 4 \)

\[
x(t+1) = bx(t)(1-x(t))
\]
Some known facts

Look at cobweb for the composite $f^3(x)$...

There exist 3-cycles for $b > 3.83$ (approximately)

**THEOREM** If a continuous function on an interval has a cycle of period 3, then it has cycles of all periods.

Sharkovskii ordering of the integers:

\[ 3 < 5 < 7 < \cdots < 2\cdot3 < 2\cdot5 < 2\cdot7 < \cdots < 2^3 \cdot 3 < 2^3 \cdot 5 < 2^3 \cdot 7 < \cdots \]

\[ \cdots < 2^n \cdot 3 < 2^n \cdot 5 < 2^n \cdot 7 < \cdots < 2^3 < 2^2 < 2 < 1 \]

**THEOREM** (Sharkovskii, 1964) Suppose a continuous map on an interval has a $p$-cycle. If $p < q$ then the map has a $q$-cycle.

**THEOREM** (Li & Yorke, 1975) Suppose

\[ f : I \text{ continuous} \rightarrow I = \text{interval} \subseteq \mathbb{R}^1 \]

has a 3-cycle.

Then there exist uncountably many aperiodic orbits (and they have sensitivity to initial conditions).

aperiodic means "not asymptotically periodic"

\[ f(x) = bx(1 - x) \]

Fixed points of composite are 2-cycles

\[ f^{23}(x) = f(f(x)) = b(bx(1 - x))(1 - bx(1 - x)) \]

\[ f^{23}(x) = f(f(x)) = b(bx(1 - x))(1 - bx(1 - x)) \]

\[ b = 2.75 \]

\[ b = 3.0 \]
\[ f(x) = bx(1-x) \]

Fixed points of composite are 2-cycles

\[ f^{(2)}(x) = f(f(x)) = b(bx(1-x))(1-bx(1-x)) \]

A period doubling cascade occurs in the box

A chaotic Cantor set lies in the box

This occurs for \( b > 3.6786 \)