LECTURE 10: Topics in Chaos

**DEFINITION:** An invariant set $A$ is an attractor if there exists an open set $U \supset A$ such that
(a) the omega limit set of each $x \in U$ lies in $A$
(b) no proper subset of $A$ has this property

Any LAS equilibrium or cycle is an attractor
The equilibria of the quadratic map for $b < 3$
The $2^n$-cycles of the quadratic map for $3 < b < 3.57$
The chaotic Cantor set $\Lambda$ of the quadratic map for $b > 4$ is not an attractor on $[0,1]$
The interval $[0,1]$ is a (“chaotic”) attractor of the quadratic map for $b = 4$

Same is true of the Tent map
The quadratic map for $3.57 < b < 4$ ??

**EXAMPLE:** The W-map
On the subinterval $I = [0.25, 0.75]$ 
W-map is same as the Tent map
$\Rightarrow I$ is an invariant chaotic set
$[0, 0.25] \cup [0.75, 1] \rightarrow I$
$\Rightarrow I$ is a chaotic attractor

Some 2D Examples
1. The Smale Horseshoe

Points that stay in square after $n = 1$ step
\[ \Lambda_0 = R_{00} \cup R_{01} \cup R_{10} \cup R_{11} \]

Points that stay in square after \( n = 2 \) steps

\( \Lambda \) is invariant under the map
All other points leave the square (\( \Lambda \) is a “repellor”)

The horseshoe map is chaotic on \( \Lambda \)!

This is proved in a way analogous to the same result for the Cantor sets of the quadratic map with \( b > 4 \)

A conjugacy is established with the shift map
on the double sequence space \( \Sigma_2 \)
by associating a point with its itinerary
with respect to the strips \( R_{i0} \) and \( R_{i1} \)

2. A circle map
\[ f(\theta) = 2\theta \]
defines a map of the unit circle \( C \) onto itself
Show \( f : C \to C \) is chaotic

Hint: Calculate the periodic points and show they are dense.
Show transitivity by looking at arcs on the circle.

Define a map on the plane by
\[ f(r, \theta) = \begin{cases} (r^{1/2}, 2\theta), & r \neq 0 \\ (0, 0), & r = 0 \end{cases} \]

The origin is an unstable equilibrium (repellor)
Since \( r(t) \to 1 \), all other orbits tend to \( C \)
\( C \) is a chaotic attractor

Continue iterating the map and obtain
\[ R_{i,j,n} = \text{horizontal strips} \]
\[ A_n = \bigcup_{i,j} R_{i,j,n} = \text{points that stay in square for } n \text{ steps} \]
\[ \Lambda_n = \bigcap_{n=1}^{\infty} A_n = \text{points that stay in square } \forall n \geq 1 \]

\( \Lambda_n \) is a Cantor set of horizontal line segments in the square

A similar analysis in reverse time produces
a Cantor set \( \Lambda \) of vertical line segments in the square
whose points stay in the square for all \( n = -1, -2, -3, \cdots \)

\( \Lambda = \Lambda_1 \cap \Lambda_2 \) is a Cantor set points
that remain in the square for all \( n = \pm 1, \pm 2, \cdots \)
3. The Henon map

\[ \begin{align*}
x(t+1) &= 1 - ax^2(t) + y(t) \\
y(t+1) &= bx(t)
\end{align*} \]

\[ a, b \in \mathbb{R}^1, \ |b| < 1 \]

The Henon map is invertible (one-to-one)

Equilibria:

\[ \begin{align*}
x_* &= \frac{1}{2a} \left( b - 1 \pm \sqrt{b^2 - 1} + 4a \right) \\
y_* &= \frac{b}{2a} \left( b - 1 \pm \sqrt{b^2 - 1} + 4a \right)
\end{align*} \]

(1) With \( a \) as a bifurcation parameter, a saddle-node bifurcation occurs at

\[ a_c = \frac{3}{4} (1-b)^2 \]

Exercises in Linearization:

(2) \( x_* \) is a saddle and \( x_* \) is LAS for \( a < a_c \)

(3) \( x_* \) loses stability at \( a = a_c \) because an eigenvalue of the Jacobian passes through \( \lambda = -1 \)
THEOREM (Devaney & Nitecki, 1979)
Let \( H \) denote the Henon map and assume
\[
a \geq (5 + 2\sqrt{5})(1 + |b|^2) / 4, \quad b \neq 0.
\]
Let \( S = \{ (x, y) : |x| \leq L, |y| \leq L \} \) where
\[
L = \left( 1 + |b| + \sqrt{(1 + |b|)^2 + 4a} \right) / 2.
\]
Define \( \Lambda = \bigcap_{n=-\infty}^{\infty} H^{n}(S) \).
Then
(1) \( \Lambda \) is a Cantor set
(2) \( H \) is conjugate to \( \Sigma \) and hence is chaotic on \( \Lambda \).

In 2D or higher dimensional maps other kinds
of bifurcation "routes to chaos" can occur.

4. The LPA Model
\[
L(t+1) = bL(t) \exp(-c_{pa}L(t) - c_{pa}A(t))
\]
\[
P(t+1) = (1 - \mu_{l})L(t)
\]
\[
A(t+1) = P(t) \exp\left(-c_{pa}A(t) + (1 - \mu_{l})A(t)\right)
\]
\[
\mu_{l} = 0.2, \quad c_{pa} = c_{ma} = 0.1
\]

\[
\begin{array}{c}
\text{Total population size} = L + P + A \\
\text{Time}
\end{array}
\]
**Liapunov Exponents**

**DEFINITION:** \( f : X \rightarrow X \) (a metric space) has sensitivity to initial conditions (SIC) if:

\[
\exists \delta > 0 \text{ such that given } x \in X \text{ and any open } U \subseteq X, x \in U
\]

\[
\exists y \in U \text{ for which } d\left(f^m(x), f^m(y)\right) > \delta \text{ for some integer } m
\]

**Example:** linear repellor

\[
x(t) = a'x(0), a > 1
\]

\[
|x(t) - y(t)| = a'|x(0) - y(0)|
\]

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**How to “recover” \( a \) from an orbit?**

\[
\delta x = x(0) - y(0) = \text{“error” in initial condition}
\]

**error after \( t \) steps:**

\[
e_t = a'\left(x_0 + \delta x\right) - a'x_0 = a'\delta x
\]

**relative error after \( t \) steps:**

\[
e_t = \frac{a'\delta x}{\delta x} = a'
\]

\[
\ln|\delta x| = -\frac{1}{t} \ln\left|\frac{e_t}{e_0}\right| \text{ for all } t, \delta x
\]

As the initial error decreases:

\[
\ln|\delta x| = \lim_{t \to \infty} \frac{1}{t} \ln\left|\frac{e_t}{e_0}\right|
\]

As the elapsed time increases:

\[
\ln|\delta x| = \lim_{t \to \infty} \frac{1}{t} \ln\left|\frac{e_t}{e_0}\right|
\]

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**Sensitivity to initial conditions if \(|\delta x| > 1\) or**

\[
\lim_{t \to \infty} \frac{1}{t} \ln\left|\frac{e_t}{e_0}\right| > 0
\]

**In general**

\[
x(t+1) = f(x(t))
\]

As a diagnostic quantity for SIC consider:

\[
\lambda(x) = \lim_{t \to \infty} \frac{1}{t} \ln\left|\frac{e_t}{e_0}\right|
\]

\[
\lambda(x) = \lim_{t \to \infty} \frac{1}{t} \ln\left|\frac{f^{m+1}(x_0 + \delta x) - f^m(x_0)}{\delta x}\right|
\]
The Liapunov exponent associated with an orbit is
\[ \lambda(x_0) = \lim_{t \to \infty} \frac{1}{t} \sum_{i=0}^{t-1} |f'(x(i))| \]
The Lyapunov exponent associated with an orbit is
\[ \lambda(x_0) = \lim_{t \to \infty} \frac{1}{t} \sum_{i=0}^{t-1} \ln |f'(x(i))| \]

The orbit has SIC if \( \lambda(x_0) > 0 \)
If the orbit is dense in an attractor we say the attractor has SIC if the orbit has SIC.

**EXAMPLE** (rotation of the circle)
Let \( C \) be the unit circle in the complex plane
Define \( f : C \to C \) by \( f(x) = xe^{2\pi i} \)
\[ f'(x) = e^{2\pi i} \]
\[ \lambda(x_e) = \lim_{t \to \infty} \frac{1}{t} \sum_{i=0}^{t-1} \ln |f'(x_e)| = \lim_{t \to \infty} \frac{1}{t} \sum_{i=0}^{t-1} \ln 1 = 0 \]
Does not have SIC

**EXAMPLE**
For the Tent map \( T(x) \)
\[ |T'(x)| = 2 \]
\[ \Rightarrow \lambda(x_0) = \lim_{t \to \infty} \frac{1}{t} \sum_{i=0}^{t-1} \ln 2 \]
\[ = \lim_{t \to \infty} \frac{1}{t} \ln 2 \]
\[ = \ln 2 \approx 0.693 \]

\( > 0 \)

The Liapunov exponent of a stable equilibrium is negative.
The same is true of a stable cycle.

**EXAMPLE**
For suppose \( x_e \) is an equilibrium of
\[ f(t + 1) = f(x(t)) \]
for which \( |f'(x_e)| < 1 \)
\[ \lambda(x_e) = \lim_{t \to \infty} \frac{1}{t} \sum_{i=0}^{t-1} \ln |f'(x_e)| = \lim_{t \to \infty} \frac{1}{t} \ln |f'(x_e)| \]
\[ \lambda(x_e) = \ln |f'(x_e)| < 0 \]
The Liapunov exponent of a stable equilibrium is negative.
The same is true of a stable cycle.
Higher Dimensional Maps

\[ x(t+1) = f(x(t)), \quad x(0) = x_0 \]

\[ f : \mathbb{R}^n \to \mathbb{R}^m \]

\[ J^{(1)}(x_0) = \text{Jacobian of } f^{(1)} \text{ evaluated at } x_0 \]

\[ U(x_0) = \text{unit sphere centered at } x_0 \]

\[ J^{(1)}(x_0)U = \text{ellipsoid} \]

\[ r^{(1)} = \text{length of the longest orthogonal axis of } J^{(1)}(x_0)U \]

**DEFINITION:** The dominant Liapunov exponent of \( x_0 \) is

\[ \lambda_1(x_0) = \lim_{t \to \infty} \frac{1}{t} \ln r^{(1)} \]

**EXAMPLE**

Define a map on the plane \( \mathbb{R}^2 \) by

\[ f(r, \theta) = \left( \frac{r^{(1/2)}}{2\theta} \right) \]

\[ r(t+1) = r^{1/2}(t) \]

\[ \theta(t+1) = 2\theta(t) \]

\[ r(0) = 1, \quad \theta(0) = 0 \]

Orbit remains on the unit circle (and is chaotic by a previous Example)

Equations are uncoupled, Jacobian is diagonal, and so we can calculate LE’s from two 1D equations

\[ \lambda_1 = \lim_{t \to \infty} \frac{1}{t} \sum_{i=0}^{i\leq t} \ln \left| \frac{d(2\theta)}{d\theta} \right| = \lim_{t \to \infty} \frac{1}{t} \sum_{i=0}^{i\leq t} \ln 2 = \ln 2 > 0 \]

\[ \lambda_2 = \lim_{t \to \infty} \frac{1}{t} \sum_{i=0}^{i\leq t} \ln \left| \frac{d(r^{1/2})}{dr} \right| = \lim_{t \to \infty} \frac{1}{t} \sum_{i=0}^{i\leq t} \ln \frac{1}{2} = -\ln 2 < 0 \]

\( \lambda_1 > 0 \) is an indication that the orbit has SIC (we knew this anyway, since we proved the orbit is chaotic in a previous example)

\( \lambda_2 < 0 \) is an indication that the circle is attracting (i.e., is a chaotic attractor)
Define a map on the plane $\mathbb{R}^2$ by $f(r, \theta) = \left( \begin{array}{c} r^2 \\ \theta + \alpha \end{array} \right)$

$r(t + 1) = r^2(t)$

$\theta(t + 1) = \theta(t) + \alpha, \quad \alpha > 0$

$r(0) = 1, \quad \theta(0) = 0$

$\alpha$ is irrationally related to $2\pi$

Orbit remains on the unit circle (and is dense)

Calculations similar to those in previous Example yield

$\lambda_1 = \ln 2 > 0, \quad \lambda_2 = 0$

Liapunov exponents are often numerically estimated (using computer programs) to help determine the possibility of chaos

EXAMPLE

In previous two Examples the unit circle is an invariant loop

In general, 0 is a Liapunov exponent of an invariant loop

In the first Example the invariant loop is a chaotic attractor.

In the second Example the invariant loop

✓ has a dense (quasiperiodic) orbit
✓ has SIC
✓ is not chaotic
✓ is not an attractor

EXAMPLE

EXAMPLE (the LPA Model)