Sequences and Series

Algebra 2
Chapter 12

Algebra II 12
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Some examples and diagrams are taken from the textbook.

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Sequence
- Function whose domain are integers
- List of numbers that follow a rule

- 2, 4, 6, 8, 10
  - Finite
- 2, 4, 6, 8, 10, ...
  - Infinite

n is like x, a_n is like y
12.1 Define and Use Sequences and Series

Rule
- \(a_n = 2n\)

Domain: (n)
- Term’s location (1\(^{st}\), 2\(^{nd}\), 3\(^{rd}\)...)

Range: (\(a_n\))
- Term’s value (2, 4, 6, 8...)

Writing rules for sequences

Look for patterns

Guess-and-check

\[ \frac{2}{5}, \frac{2}{25}, \frac{2}{125}, \frac{2}{625}, \ldots \]

\[ a_n = \frac{2}{5^n} \]

\[ 2(1)+1, 2(2)+1, 2(3)+1, \ldots \Rightarrow a_n = 2n + 1 \]
The n's are integers so there is no values between the integers.
12.1 Define and Use Sequences and Series

Series

Sum of a sequence

2, 4, 6, 8, ... → sequence

2 + 4 + 6 + 8 + ... → series
12.1 Define and Use Sequences and Series

- **Sigma notation**
  - **Finite**
    \[ 2 + 4 + 6 + 8 = \sum_{i=1}^{4} 2i \]
  - **Infinite**
    \[ 2 + 4 + 6 + 8 + \cdots = \sum_{i=1}^{\infty} 2i \]

- **Upper limit**
- **Lower limit**
- **Index of summation (variable)**
\[
\sum_{n=1}^{25} 4n
\]

\[
\sum_{n=1}^{\infty} \frac{n+1}{n^2}
\]

Note that the index may be any letter.
Find the sum of the series

\[ \sum_{k=5}^{10} k^2 + 1 \]

\[ 5^2 + 1 + 6^2 + 1 + 7^2 + 1 + 8^2 + 1 + 9^2 + 1 + 10^2 + 1 = 361 \]
Some shortcut formulas

\[ \sum_{i=1}^{n} 1 = n \]
\[ \sum_{i=1}^{n} i = \frac{n(n + 1)}{2} \]
\[ \sum_{i=1}^{n} i^2 = \frac{n(n + 1)(2n + 1)}{6} \]
Find the sum of the series

$$\sum_{k=1}^{10} 3k^2 + 2$$

798 #3-63 every other odd, 65 + 3 = 20

$$3 \frac{n(n + 1)(2n + 1)}{6} + 2n = 3 \frac{10(10 + 1)(2(10) + 1)}{6} + 2(10) = 1175$$
12.1 Homework Quiz
12.2 Analyze Arithmetic Sequences and Series

- Arithmetic Sequences
  - Common difference (d) between successive terms
  - Add the same number each time
  - 3, 6, 9, 12, 15, ...
  - d = 3

- Is it arithmetic?
  - -10, -6, -2, 0, 2, 6, 10, ...
  - 5, 11, 17, 23, 29, ...

No
Yes, d = 6
Formula for $n^{th}$ term

$a_n = a_1 + (n - 1)d$

Write a rule for the $n^{th}$ term

32, 47, 62, 77, ...

\[ d = 15 \]

\[ a_n = 32 + (n-1)15 = 32 + 15n - 15 \Rightarrow a_n = 17 + 15n \]
12.2 Analyze Arithmetic Sequences and Series

One term of an arithmetic sequence is $a_8 = 50$. The common difference is 0.25. Write the rule for the $n^{th}$ term.

\[ a_n = a_1 + (n - 1)d \]
\[ 50 = a_1 + (8 - 1)0.25 \rightarrow 50 = a_1 + 1.75 \rightarrow 48.25 = a_1 \]
\[ a_n = 48.25 + (n - 1)0.25 \rightarrow a_n = 48.25 + 0.25n - 0.25 \rightarrow a_n = 48 + 0.25n \]
Two terms of an arithmetic sequence are \( a_5 = 10 \) and \( a_{30} = 110 \). Write a rule for the \( n \)th term.

\[
a_n = a_1 + (n - 1)d
\]

\[
10 = a_1 + (5 - 1)d \quad \rightarrow \quad 10 = a_1 + 4d
\]

\[
110 = a_1 + (30 - 1)d \quad \rightarrow \quad 110 = a_1 + 29d
\]

Linear combination

\[
-10 = -a_1 - 4d
\]

\[
110 = a_1 + 29d
\]

\[
100 = 25d
\]

\[
d = 4
\]

Substitute

\[
10 = a_1 + 4d \rightarrow 10 = a_1 + 4(4) \rightarrow a_1 = -6
\]

Rule

\[
a_n = -6 + (n - 1)4 \rightarrow a_n = -6 + 4n - 4 \rightarrow a_n = 4n - 10
\]
From example:
First and last ($a_1 + a_n$) = 11
10 numbers but only half as many pairs ($n/2$)
Consider the arithmetic series

\[ 20 + 18 + 16 + 14 + \ldots \]

Find the sum of the first 25 terms.

\[
a_{25} = 20 + (25-1)(-2) = -28 \\
S_{25} = 25((20+28)/2) = -100
\]
12.2 Analyze Arithmetic Sequences and Series

- Consider the arithmetic series
  - $20 + 18 + 16 + 14 + \cdots$
- Find $n$ such that $S_n = -760$

- $806 \#3-63$ every other odd, $65 + 3 = 20$

$$a_n = 20 + (n-1)(-2) = 22 - 2n$$

$$S_n = -760 = n((20 + 22 - 2n)/2) \rightarrow -1520 = n(42 - 2n) \rightarrow -1520 = 42n - 2n^2 \rightarrow 2n^2 - 42n - 1520 = 0 \rightarrow n^2 - 21n - 760 = 0 \rightarrow (n+19)(n-40) = 0 \rightarrow n = 40, -19$$
12.2 Homework Quiz
12.3 Analyze Geometric Sequences and Series

- Created by multiplying by a common ratio (r)
- Are these geometric sequences?
  - 1, 2, 6, 24, 120, ...
  - 81, 27, 9, 3, 1, ...

No
Yes r = 1/3
$r = \frac{2}{5}$

$a_n = 5 \left(\frac{2}{5}\right)^{n-1}$

$a_8 = 5 \left(\frac{2}{5}\right)^7 = 0.008192$
One term of a geometric sequence is $a_4 = 3$ and $r = 3$. Write the rule for the $n^{th}$ term.

$$a_4 = 3 = a_1 \cdot 3^{4-1} \implies 3 = a_1 \cdot 27 \implies a_1 = \frac{1}{9}$$

$$a_n = (1/9) \cdot 3^{n-1}$$
12.3 Analyze Geometric Sequences and Series

If two terms of a geometric sequence are $a_2 = -4$ and $a_6 = -1024$, write rule for the $n^{th}$ term.

\[ a_2 = -4 = a_1 r^{2-1} \rightarrow -4 = a_1 r \]
\[ a_6 = -1024 = a_1 r^{6-1} \rightarrow -1024 = a_1 r^5 \]

Solve first for $a_1$: $a_1 = -4/r$
Plug into second: $-1024 = (-4/r)r^5 \rightarrow -1024 = -4r^5/r \rightarrow -1024 = -4r^4 \rightarrow 256 = r^4 \rightarrow r = 4$
Plug back into first: $a_1 = -4/4 \rightarrow a_1 = -1$
Write rule: $a_n = -1 \cdot 4^{n-1}$
r = \frac{1}{2}, a_1 = 4

S_{10} = 4\left(\frac{1 - \left(\frac{1}{2}\right)^{10}}{1 - \frac{1}{2}}\right) = 4(0.99902/0.5) = 7.992 = \frac{1023}{128}
Find $n$ such that $S_n = 31/4$

4 + 2 + 1 + ½ + …

814 #3-47 every other odd, 49, 51, 53, 57, 59 + 3 = 20

31/4 = 4((1 – (1/2)^n)/(1-1/2)) → 31/16 = (1 – (1/2)^n)/(1/2) → 31/32 = 1 – (1/2)^n → -1/32 = -(1/2)^n → 1/32 = (1/2)^n → log (1/32) = n log (1/2) → n = (log (1/32)/log (1/2)) = 5
12.3 Homework Quiz
Think of the box a 1 whole piece
12.4 Find the Sums of Infinite Geometric Series

Cut in half
12.4 Find the Sums of Infinite Geometric Series

Cut in half
12.4 Find the Sums of Infinite Geometric Series

\[ \frac{1}{2} \]

\[ \frac{1}{4} \]

\[ \frac{1}{2} + \frac{1}{4} \]
12.4 Find the Sums of Infinite Geometric Series

\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{8}
\]
12.4 Find the Sums of Infinite Geometric Series

\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}
\]

Cut in half
What is the sum of the pieces if we keep going? 1 piece
12.4 Find the Sums of Infinite Geometric Series

- Sum of an infinite geometric series
  \[ S = \frac{a_1}{1-r} \]

- \(|r| < 1\)
  - If \(|r| > 1\), then no sum (\(\infty\))
\[ a_1 = 2(0.1)^{1-1} = 2, \quad r = 0.1 \]
\[ S_\infty = \frac{2}{1 - 0.1} = \frac{2}{0.9} = \frac{20}{9} \]

\[ a_1 = 12, \quad r = \frac{1}{3} \]
\[ S = \frac{12}{1 - \frac{1}{3}} = \frac{12}{\frac{2}{3}} = \frac{36}{2} = 18 \]
12.4 Find the Sums of Infinite Geometric Series

An infinite geometric series has $a_1 = 5$ has sum of $27/5$. Find the common ratio.

$\frac{27}{5} = \frac{5}{1-r} \rightarrow 27(1-r) = 5 \cdot 5 \rightarrow 1-r = \frac{25}{27} \rightarrow -r = -\frac{2}{27} \rightarrow r = \frac{2}{27}$
Write the repeating unit as a sum of fractions
27/100 + 27/10000 + 27/1000000 + ...
\( a_1 = \frac{27}{100}, \ r = \frac{1}{100} \)
\[ S = \frac{\frac{27}{100}}{1-\left(\frac{1}{100}\right)} = \frac{27}{100}/\left(\frac{99}{100}\right) = \frac{27}{99} = \frac{3}{11} \]
\[ \frac{41}{100} + \frac{6}{1000} + \frac{6}{10000} + \frac{6}{100000} + \ldots \]

Ignore the \( \frac{41}{100} \) for now.

\[ a_1 = \frac{6}{1000}, \quad r = \frac{1}{10} \]
\[ S = \frac{(6/1000)}{(1-1/10)} = \frac{(6/1000)}{(9/10)} = \frac{60}{9000} = \frac{1}{150} \]

Now add the \( \frac{41}{100} \)
\[ \frac{41}{100} + \frac{1}{150} \rightarrow \frac{(41\times3)}{300} + \frac{(1\times2)}{300} \rightarrow \frac{123}{300} + \frac{2}{300} \rightarrow \frac{125}{300} \rightarrow \frac{5}{12} \]
12.4 Homework Quiz
Both these rules give the same sequence
Write the first 5 terms
\[ a_1 = 1, \quad a_n = (a_{n-1})^2 + 1 \]

\[ a_1 = 2, a_2 = 2, a_n = a_{n-2} - a_{n-1} \]

\[ a_1 = 1, \quad a_2 = 1^2 + 1 = 2, \quad a_3 = 2^2 + 1 = 5, \quad a_4 = 5^2 + 1 = 26, \quad a_5 = 26^2 + 1 \quad 677 \]

\[ a_1 = 2, \quad a_2 = 2, \quad a_3 = 2 - 2 = 0, \quad a_4 = 2 - 0 = 2, \quad a_5 = 0 - 2 = -2 \]
Explicit: $a_n = 15 + (n-1)5 \rightarrow a_n = 5n + 10$

Recursive: $a_1 = 15$, $a_n = a_{n-1} + 5$
Explicit: \( a_n = 4(0.2)^{n-1} \)

Recursive: \( a_1 = 4, a_n = 0.2a_{n-1} \)
Write a recursive rule for

1, 1, 4, 10, 28, 76, ...

1, 2, 2, 4, 8, 32, ...

\[ a_1 = 1, \ a_2 = 1, \ a_n = 2(a_{n-2} + a_{n-1}) \]

\[ a_1 = 1, \ a_2 = 2, \ a_n = (a_{n-2})(a_{n-1}) \]
12.5 Use Recursive Rules with Sequences and Functions

- Iterations
  - Repeated composition of functions
  - \( f(f(f(x))) \)
  - Use \( x \) to find \( f(x) \)
    - Use that value to find the next \( f(x) \)
  - \( x_1 = f(x_0); x_2 = f(x_1), \ldots \)
  - Find the first three iterations of the function.
    - \( f(x) = 4x - 3, x_0 = 2 \)

\[ x_0 = 2 \]
\[ x_1 = f(2) = 4(2) - 3 = 5 \]
\[ x_2 = f(5) = 4(5) - 3 = 17 \]
\[ x_3 = f(17) = 4(17) - 3 = 65 \]
12. Review

1. 5, 9, 13, 17, …
2. 3, 6, 12, 24, …
3. 40, 10, 2, 0.5, …
4. 4, 7, 12, 19, …

Write the first six terms of the sequence.
3. \( a_1 = 6 \), \( a_n = -a_{n-1} \)
4. \( a_1 = 1 \), \( a_n = 5a_{n-1} \)
5. \( a_1 = 3 \), \( a_n = a_{n-1} + 6 \)

Write the next term of the sequence, and then write a rule for the nth term.
6. \( a_1 = 5 \), \( a_n = 2a_{n-1} + 1 \)
7. \( a_1 = 4 \), \( a_n = a_{n-1} - 3 \)
8. \( a_1 = 1 \), \( a_n = a_{n-1} + 5 \)

Find the sum of the series.
9. \( \sum_{n=1}^{\infty} \frac{1}{n^2} \)
10. \( \sum_{n=1}^{\infty} \frac{1}{2^n} \)
11. \( \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \)
12. \( \sum_{n=1}^{\infty} \frac{1}{n^2} \)

Write the repeating decimal as a fraction in lowest terms.
13. \( 0.111… \)
14. \( 0.666… \)
15. \( 0.878787… \)
16. \( 0.352352… \)

Write a recursive rule for the sequence.
17. \( 2, 12, 22, 32, \ldots \)
18. \( 5, 9, 13, 17, \ldots \)
19. \( 1, 5, 9, 13, \ldots \)
20. \( 2, 4, 6, 8, \ldots \)

Find the first three iterates of the function for the given initial value.
21. \( f(x) = 2x - 1 \), \( x_0 = 4 \)
22. \( f(x) = 1 - 2x \), \( x_0 = 1 \)
23. \( f(x) = x^2 + 2 \), \( x_0 = -1 \)

QUILTS Use the pattern of checkerboard quilts shown.

- What does \( n \) represent for each quilt? What does \( a_n \) represent?
- Make a table that shows \( n \) and \( a_n \) for \( n = 1, 2, 3, 4, 5, 6, 7, \) and 8.
- Use the rule \( a_n = a_{n-1} + 1 \) to find \( a_n \) for \( n = 1, 2, 3, 4, 5, 6, 7, \) and 8. Compare these values with the results in your table. What can you conclude about the sequence defined by this rule?

AUDITIONS Several rounds of auditions are being held to cast the three main parts in a play. There are 872 actors at the first round of auditions. In each successive round of auditions, one third of the actors from the previous round remain. Find a rule for the number \( a_n \) of actors in the nth round of auditions. For what values of \( n \) does your rule make sense?