**10.3 Ellipses**

**Ellipse** is the set of all points in a plane where the sum of the distances to two fixed points, **foci**, is constant.

<table>
<thead>
<tr>
<th>Horizontal Ellipse</th>
<th>Vertical Ellipse</th>
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</thead>
<tbody>
<tr>
<td><img src="image1" alt="Horizontal Ellipse Diagram" /></td>
<td><img src="image2" alt="Vertical Ellipse Diagram" /></td>
</tr>
</tbody>
</table>

**Center at** (h, k)
- **Horizontal major axis length** = 2a
- **Vertical minor axis length** = 2b
- **Vertices** (h ± a, k)
- **Vertices** (h, k ± b)
- **Foci** (h ± c, k)

\[
\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1
\]

**Center at** (h, k)
- **Vertical major axis length** = 2a
- **Horizontal minor axis length** = 2b
- **Vertices** (h, k ± a)
- **Vertices** (h ± b, k)
- **Foci** (h, k ± c)

\[
\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1
\]

- **a** is the distance from center to vertex.
- **b** is the distance from center to covertex.
- **c** is the distance from center to focus.

**Example 1:**

Find the center, vertices, and foci of the ellipse given by \(9x^2 + 4y^2 = 36\).

**Solution**

Put the equation in standard form by dividing by 36 so the equation equals 1.

\[
\frac{x^2}{4} + \frac{y^2}{9} = 1
\]

The bigger denominator is \(a^2\), so rearrange the equation.

\[
\frac{y^2}{9} + \frac{x^2}{4} = 1
\]

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Since \( y \) is above \( a^2 \), the ellipse is vertical.

\[ a^2 = 9 \text{ so } a = 3 \]
\[ b^2 = 4 \text{ so } b = 2 \]
\[ c^2 = a^2 - b^2 \text{ so } c^2 = 3^2 - 2^2. c = \sqrt{5} \]

Also, \( h = 0, k = 0 \), so the center is \((0, 0)\).

Vertices are \((0, \pm 3)\)

Covertices are \((\pm 2, 0)\)

Foci are \((0, \pm \sqrt{5})\)

**Example 2:**

Find the standard form of the equation of the ellipse centered at \((1, 2)\) with major axis length 10 and foci at \((-2, 2)\) and \((4, 2)\). Then graph it.

**Solution**

If you graph the center and the foci, you will see that the major axis through the points is horizontal, so this is a horizontal ellipse.

The center = \((1, 2) = (h, k)\).

The major axis length is \(10 = 2a\), so \(a = 5\).

\(c\) is the distance from the center to the foci, so \(c = 3\).

\[ c^2 = a^2 - b^2, \text{ so } 3^2 = 5^2 - b^2. \]

\[ b^2 = 5^2 - 3^2 = 16, \text{ so } b = 4. \]

Now write the equation.

\[
\frac{(x - 1)^2}{25} + \frac{(y - 2)^2}{16} = 1
\]

The vertices are \((h \pm a, k) = (-4, 2)\) and \((6, 2)\).

The covertices are \((h, k \pm b) = (1, -2)\) and \((1, 6)\).

Graph by plotting the vertices and covertices, then drawing an approximate ellipse.

**Example 3:**

Sketch the graph of the following ellipse:

\[ 25x^2 + 9y^2 - 200x + 36y + 211 = 0 \]

**Solution**

Complete the square by moving the constant to the other side and factoring the \(x\)'s and \(y\)'s.

\[ 25(x^2 - 8x) + 9(y^2 + 4y) = -211 \]
Add \( \left( \frac{1}{2} b \right)^2 \) for both the x's and y's. On the right side, don’t forget to multiply by the coefficient.

\[
25 \left( x^2 - 8x + \left( \frac{1}{2}(-8) \right)^2 \right) + 9 \left( y^2 + 4y + \left( \frac{1}{2}(4) \right)^2 \right) = -211 + 25 \left( \left( \frac{1}{2}(-8) \right)^2 \right) + 9 \left( \left( \frac{1}{2}(4) \right)^2 \right)
\]

\[
25(x^2 - 8x + 16) + 9(y^2 + 4y + 4) = -211 + 400 + 36
\]

Factor the left side.

\[
25(x - 4)^2 + 9(y + 2)^2 = 225
\]

Divide by 225 to make the equation equal 1.

\[
\frac{(x - 4)^2}{9} + \frac{(y + 2)^2}{25} = 1
\]

The largest denominator should come first, so rearrange the equation.

\[
\frac{(y + 2)^2}{25} + \frac{(x - 4)^2}{9} = 1
\]

y comes first, so this must be a vertical ellipse. Compare your equation to the standard form and see that \( h = 4, k = -2, a^2 = 25 \) so \( a = 5 \), and \( b^2 = 9 \) so \( b = 3 \).

Graph by plotting the center. Move vertically the distance of \( a = 5 \) to get the vertices. From the center move horizontally the distance of \( b = 3 \) to get the covertices. Sketch an ellipse through these points.

**Eccentricity**

Eccentricity is a measure of how circular an ellipse is. \( e = \frac{c}{a} \) where \( 0 < e < 1 \).

If the eccentricity is near 0, then the ellipse is almost a circle.

If the eccentricity is near 1, then the ellipse is almost a line.