**Hyperbolas** are the set of all points in a plane where the difference of the distances from two set points, **foci**, is constant.

### Horizontal Hyperbola
- **Center** at $(h, k)$
- **Horizontal transverse axis length** = $2a$
- **Vertical conjugate axis length** = $2b$
- **Vertices** $(h ± a, k)$
- **Covertices** $(h, k ± b)$
- **Foci** $(h ± c, k)$
- **Standard Form Equation**
  \[
  \frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1
  \]
- **Asymptotes**
  \[
  y = k ± \frac{b}{a}(x - h)
  \]

### Vertical Hyperbola
- **Center** at $(h, k)$
- **Vertical transverse axis length** = $2a$
- **Horizontal conjugate axis length** = $2b$
- **Vertices** $(h, k ± a)$
- **Covertices** $(h ± b, k)$
- **Foci** $(h, k ± c)$
- **Standard Form Equation**
  \[
  \frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1
  \]
- **Asymptotes**
  \[
  y = k ± \frac{a}{b}(x - h)
  \]

**Eccentricity** is $e = \frac{c}{a}$ where $e > 1$. The larger the eccentricity is, the closer the branches are to straight lines.

**Example 1:**

Find the center, vertices, asymptotes, and foci of the hyperbola given by $4y^2 - 9x^2 = 36$.

**Solution**

Write the equation in standard form by dividing by 36 so that the equation equals 1.

\[
\frac{y^2}{9} - \frac{x^2}{4} = 1
\]

Because $y$ comes first, this is a vertical hyperbola. Compare this equation with the standard form to see that $h = 0, k = 0, a^2 = 9$ so $a = 3$, and $b^2 = 4$ so $b = 2$.

\[
c^2 = a^2 + b^2
c^2 = 9 + 4 = 13
c = \sqrt{13}
\]
Center is (0, 0)
Vertices are (0, -3) and (0, 3)
Asymptotes are $y = \pm \frac{3}{2}x$
Foci are (0, $-\sqrt{13}$) and (0, $\sqrt{13}$)

Example 2:
Find the standard form of the equation of the hyperbola centered at (1, 2) with transverse axis length 10 and foci at (-5, 2) and (7, 2). Then graph it.

Solution
Graphing the three points shows that the transverse axis is horizontal.
The transverse axis length is $2a = 10$, so $a = 5$.
The distance from the center to the foci is $c$, so $c = 6$.
Find $b$

$$c^2 = a^2 + b^2$$
$$6^2 = 5^2 + b^2$$
$$b^2 = 11$$
$$b = \sqrt{11}$$

The center is (1, 2) so $h = 1$ and $k = 2$.
Write the standard form equation for a horizontal hyperbola.

$$\frac{(x - 1)^2}{25} - \frac{(y - 2)^2}{11} = 1$$

Graph it by plotting the center.
Given that it is a horizontal ellipse, move the distance $a = 5$ horizontally from the center and a distance $b = \sqrt{11}$ vertically from the center.
Draw a rectangle with these four points as the center of each side.
Draw diagonal lines through the corners of the rectangle. These are the asymptotes.
Sketch the hyperbola starting near an asymptote, curving through the vertex, and ending near the other asymptote.
Example 3:

Sketch the graph of the hyperbola given by

\[ 4x^2 - 9y^2 - 24x - 72y - 72 = 0 \]

Solution

Write the equation in standard form by completing the square. Start by moving the constant to the other side and factoring the x's and y's.

\[ 4(x^2 - 6x) - 9(y^2 - 8y) = 72 \]

Add \( \left( \frac{1}{2}b \right)^2 \) for both the x's and y's. On the right side, don't forget to multiply by the coefficient.

\[
\begin{align*}
4 \left( x^2 - 6x + \left( \frac{1}{2} \cdot -6 \right)^2 \right) - 9 \left( y^2 + 8y + \left( \frac{1}{2} \cdot 8 \right)^2 \right) &= 72 + 4 \left( \frac{1}{2} \cdot (-6) \right)^2 - 9 \left( \frac{1}{2} \cdot 8 \right)^2 \\
4(x^2 - 6x + 9) - 9(y^2 + 8y + 16) &= 72 + 36 - 144 \\
\end{align*}
\]

Factor the left side.

\[ 4(x - 3)^2 - 9(y + 4)^2 = -36 \]

Divide both sides by -36.

\[ \frac{(x - 3)^2}{9} - \frac{(y + 4)^2}{4} = 1 \]

Rearrange.

\[ \frac{(y + 4)^2}{4} - \frac{(x - 3)^2}{9} = 1 \]

This is a vertical hyperbola.

The center is at (3, -4), \( a = 2 \) and \( b = 3 \).

Go vertically \( a = 2 \) from the center and horizontally \( b = 3 \) from the center. These four points form the box. Draw the diagonals and then the hyperbola.

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General Equations of Conics

All vertical and horizontal conics can be written in this general form.

\[ Ax^2 + Cy^2 + Dx + Ey + F = 0 \]

- Circle if \( A = C \)
- Parabola if \( AC = 0 \) (so \( A = 0 \) or \( C = 0 \))
- Ellipse if \( AC > 0 \)
- Hyperbola if \( AC < 0 \)
Example 5:

Classify each conic

a) \(4x^2 + 5y^2 - 9x + 8y = 0\)
b) \(2x^2 - 5x + 7y - 8 = 0\)
c) \(7x^2 + 7y^2 - 9x + 8y - 16 = 0\)
d) \(4x^2 - 5y^2 - x + 8y + 1 = 0\)

Solution

a) \(AC = 4(5) = 20\) so Ellipse
b) \(AC = 2(0) = 0\) so Parabola
c) \(A = C = 7\) so Circle
d) \(AC = 4(-5) = -20\) so Hyperbola