ANALYTICAL GEOMETRY IN THREE DIMENSIONS

PRECALCULUS
CHAPTER 11
This Slideshow was developed to accompany the textbook

- Precalculus with Limits
- By Larson, R., Hostetler, R.
- 2007 Houghton Mifflin Company

Some examples and diagrams are taken from the textbook.
11.1 3-D COORDINATE SYSTEM

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11.1 3-D COORDINATE SYSTEM

• Points in 3 dimensions
  • \((x, y, z)\)
  • \(x\) comes out/into of paper
  • \(y\) is left/right
  • \(z\) is up/down

• Graph by moving out the \(x\), over the \(y\), then up the \(z\).
  • Graph A\((5, 6, 3)\)
  • Graph B\((-2, -4, 0)\)
11.1 3-D COORDINATE SYSTEM

• Distance Formula
  • In 2-D:
    • \( d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \)
  • In 3-D: (just add the \( z \))
    • \( d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \)
11.1 3-D COORDINATE SYSTEM

- Equation of Circle (2-D)
  \[(x - h)^2 + (y - k)^2 = r^2\]
- Equation of Sphere (3-D) (just add z)
  \[(x - h)^2 + (y - k)^2 + (z - j)^2 = r^2\]
- Center is \((h, j, k)\), \(r = \) radius
- Graph by plotting the center and moving each direction the radius
- Graph
  \[(x - 2)^2 + (y + 1)^2 + (z + 1)^2 = 16\]
- Center \((2, -1, -1)\)
- \(r^2 = 16\) so \(r = 4\)
11.1 3-D COORDINATE SYSTEM

• Trace (like intercepts for a sphere)
  • Draw the $xy$ trace for
    $(x - 2)^2 + (y + 1)^2 + (z + 1)^2 = 16$
  • Since $xy$ trace, let $z = 0$
    $(x - 2)^2 + (y + 1)^2 + (1)^2 = 16$
    $(x - 2)^2 + (y + 1)^2 = 15$
  • Center (2, -1)
  • $r = \sqrt{15} \approx 3.9$
  • Looks funny because a perspective drawing
11.2 VECTORS IN SPACE

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11.2 VECTORS IN SPACE

- Vectors in 2-D
  - \( \vec{v} = \langle v_1, v_2 \rangle \)

- Vectors in 3-D (just add \( z \))
  - \( \vec{v} = \langle v_1, v_2, v_3 \rangle \)

- To find a vector from the initial point \( (p_1, p_2, p_3) \) to the terminal point \( (q_1, q_2, q_3) \)
  - \( \vec{v} = \langle q_1 - p_1, q_2 - p_2, q_3 - p_3 \rangle \)

- If \( \vec{v} = \langle v_1, v_2, v_3 \rangle \) and \( \vec{u} = \langle u_1, u_2, u_3 \rangle \),
  - Addition
    - Add corresponding elements
    - \( \vec{v} + \vec{u} = \langle v_1 + u_1, v_2 + u_2, v_3 + u_3 \rangle \)
  - Scalar multiplication
    - Distribute
    - \( c\vec{v} = \langle cv_1, cv_2, cv_3 \rangle \)
  - Dot Product
    - \( \vec{v} \cdot \vec{u} = v_1u_1 + v_2u_2 + v_3u_3 \)
  - Magnitude
    - \( ||\vec{v}|| = \sqrt{v_1^2 + v_2^2 + v_3^2} \)
  - Unit vector in the direction of \( \vec{v} \)
    - \( \frac{\vec{v}}{||\vec{v}||} \)
11.2 VECTORS IN SPACE

- Angle between vectors
  - $\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}|| ||\mathbf{v}|| \cos \theta$

- If $\theta = 90^\circ$ (and $\mathbf{u} \cdot \mathbf{v} = 0$)
  - Then vectors are orthogonal

- If $\mathbf{u} = c\mathbf{v}$
  - Then vectors are parallel
11.2 VECTORS IN SPACE

• Let \( \vec{m} = \langle 1, 0, 3 \rangle \) and \( \vec{n} = \langle -2, 1, -4 \rangle \)
• Find \( ||\vec{m}|| \)
  \[ ||\vec{m}|| = \sqrt{m_1^2 + m_2^2 + m_3^2} \]
  \[ = \sqrt{1^2 + 0^2 + 3^2} \]
  \[ = \sqrt{10} \]

• Find unit vector in direction of \( \vec{m} \)
  \[ \frac{\vec{m}}{||\vec{m}||} = \frac{\langle 1,0,3 \rangle}{\sqrt{10}} \]
  \[ = \langle \frac{1}{\sqrt{10}}, 0, \frac{3}{\sqrt{10}} \rangle \]

• Find \( \vec{m} + 2\vec{n} \)
  \[ \langle 1, 0, 3 \rangle + 2\langle -2, 1, -4 \rangle \]
  \[ = \langle 1, 0, 3 \rangle + \langle -4, 2, -8 \rangle \]
  \[ = \langle -3, 2, -5 \rangle \]
11.2 VECTORS IN SPACE

• Let \( \vec{m} = \langle 1, 0, 3 \rangle \) and \( \vec{n} = \langle -2, 1, -4 \rangle \)

• Find \( \vec{m} \cdot \vec{n} \)
  • \( \langle 1, 0, 3 \rangle \cdot \langle -2, 1, -4 \rangle \)
  • \( 1(-2) + 0(1) + 3(-4) \)
  • -14

• Find the angle between \( \vec{m} \) and \( \vec{n} \)
  • \( \vec{m} \cdot \vec{n} = ||\vec{m}|| ||\vec{n}|| \cos \theta \)
  • \( -14 = \sqrt{1^2 + 0^2 + 3^2} \sqrt{(-2)^2 + 1^2 + (-4)^2} \cos \theta \)
  • \( -14 = \sqrt{10} \sqrt{21} \cos \theta \)
  • \( \frac{-14}{\sqrt{10} \sqrt{21}} = \cos \theta \)
  • \( \theta \approx 165.0^\circ \)
11.2 VECTORS IN SPACE

• Are \( \vec{p} = \langle 1, 5, -2 \rangle \) and \( \vec{q} = \left\langle -\frac{1}{5}, -1, \frac{2}{5} \right\rangle \) parallel, orthogonal, or neither?

• Orthogonal if \( \vec{p} \cdot \vec{q} = 0 \)

• \( \langle 1, 5, -2 \rangle \cdot \left\langle -\frac{1}{5}, -1, \frac{2}{5} \right\rangle \)

• \( 1 \left( -\frac{1}{5} \right) + 5(-1) + (-2) \left( \frac{2}{5} \right) \)

• \( -\frac{1}{5} - 5 - \frac{4}{5} = -6 \)

• Not 0, so not orthogonal

• Parallel if \( \vec{p} = c \vec{q} \)

• \( \langle 1, 5, -2 \rangle = c \left\langle -\frac{1}{5}, -1, \frac{2}{5} \right\rangle \)

• Check \( x \)

  • \( 1 = c \left( -\frac{1}{5} \right) \rightarrow c = -5 \)

• Check \( y \)

  • \( 5 = c(-1) \rightarrow c = -5 \)

• Check \( z \)

  • \( -2 = c \left( \frac{2}{5} \right) \rightarrow c = -5 \)

• \( c \) is always the same, so they are parallel
11.2 VECTORS IN SPACE

• Are \( P(1, -1, 3) \), \( Q(0, 4, -2) \), and \( R(6, 13, -5) \) collinear?

• Find \( \overrightarrow{PQ} \) and \( \overrightarrow{QR} \). If they are parallel, then they go in same direction.

• Since they would share a point, then they would be the same line.

\[
\overrightarrow{PQ} = \langle 0 - 1, 4 - (-1), -2 - 3 \rangle = \langle -1, 5, -5 \rangle
\]

\[
\overrightarrow{QR} = \langle 6 - 0, 13 - 4, -5 - (-2) \rangle = \langle 6, 9, -3 \rangle
\]

• These are not parallel because \( \overrightarrow{PQ} \neq c\overrightarrow{QR} \)

• They are not going same direction, so not collinear
11.3 CROSS PRODUCTS

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11.3 CROSS PRODUCTS

• \( \hat{i} \) is unit vector in \( x \), \( \hat{j} \) is unit vector in \( y \), and \( \hat{k} \) is unit vector in \( z \)

• \( \mathbf{u} = u_1 \hat{i} + u_2 \hat{j} + u_3 \hat{k} \) and \( \mathbf{v} = v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k} \)

• \( \mathbf{u} \times \mathbf{v} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
u_1 & u_2 & u_3 \\
v_1 & v_2 & v_3
\end{vmatrix} \)

• If \( \mathbf{u} = \langle -2, 3, -3 \rangle \) and \( \mathbf{v} = \langle 1, -2, 1 \rangle \), find \( \mathbf{u} \times \mathbf{v} \)

• \( \mathbf{u} \times \mathbf{v} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
-2 & 3 & -3 \\
1 & -2 & 1
\end{vmatrix} \)

• \( = 3\hat{i} + (-3) \hat{j} + 4\hat{k} - 3\hat{k} - 6\hat{i} - (-2)\hat{j} \)

• \( = -3\hat{i} - \hat{j} + \hat{k} = \langle -3, -1, 1 \rangle \)
11.3 CROSS PRODUCTS

- Properties of Cross Products
- \( \mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u}) \)
- \( \mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w}) \)
- \( c(\mathbf{u} \times \mathbf{v}) = c\mathbf{u} \times \mathbf{v} = \mathbf{u} \times c\mathbf{v} \)
- \( \mathbf{u} \times \mathbf{u} = 0 \)
  - If \( \mathbf{u} \times \mathbf{v} = 0 \), then \( \mathbf{u} \) and \( \mathbf{v} \) are parallel
- \( \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} \)
- \( \mathbf{u} \times \mathbf{v} \) is orthogonal to \( \mathbf{u} \) and \( \mathbf{v} \)
- \( \|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\| \sin \theta \)
11.3 CROSS PRODUCTS

• $A = bh$
• $h = ||\vec{u}|| \sin \theta$
• $A = ||\vec{v}|| ||\vec{u}|| \sin \theta$
• Area of a Parallelogram
  • $||\vec{u} \times \vec{v}||$ where $\vec{u}$ and $\vec{v}$ represent adjacent sides
11.3 CROSS PRODUCTS

- **Triple Scalar Product (shortcut)**
  
  \[ \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \]

- **Volume of Parallelepiped**
  
  - (3-D parallelogram)
  
  \[ V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| \] where \( \mathbf{u}, \mathbf{v}, \) and \( \mathbf{w} \) represent adjacent edges
11.4 LINES AND PLANES IN SPACE

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11.4 LINES AND PLANES IN SPACE

- Lines
  - Line \( L \) goes through points \( P \) and \( Q \)
  - \( \mathbf{v} \) is a direction vector for \( L \)
  - Start at \( P \) and move any distance in direction \( \mathbf{v} \) to get some point \( Q \)
  - \( \overrightarrow{PQ} = tv \) because they are parallel
  - \( \langle x - x_1, y - y_1, z - z_1 \rangle = \langle at, bt, ct \rangle \)
  - General form
11.4 LINES AND PLANES IN SPACE

• Parametric Equations of Line
  • Take each component of the general form and solve for \( x, y, \) or \( z \).
    \[
    x = at + x_1 \\
    y = bt + y_1 \\
    z = ct + z_1
    \]
  • We used these when we solved 3-D systems of equations and got many solutions

• Symmetric Equation of Line
  • Solve each equation in parametric equations for \( t \)
    \[
    \frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}
    \]
11.4 LINES AND PLANES IN SPACE

- Find a set of parametric equations of the line that passes through \((1, 3, -2)\) and \((4, 0, 1)\).
- Find the direction vector between those two points.
  - \(\vec{v} = \langle 4 - 1, 0 - 3, 1 - (-2) \rangle\)
  - \(= \langle 3, -3, 3 \rangle\)
  - \(= \langle a, b, c \rangle\)
- Let’s call the first point \((1, 3, -2) = (x_1, y_1, z_1)\)
- Plug it in
  \[x = at + x_1\]
  \[y = bt + y_1\]
  \[z = ct + z_1\]
  \[x = 3t + 1\]
  \[y = -3t + 3\]
  \[z = 3t - 2\]
11.4 LINES AND PLANES IN SPACE

- Planes
- \( \vec{PQ} \cdot \vec{n} = 0 \) because they are perpendicular

- Standard form
  - \( a(x - x_1) + b(y - y_1) + c(z - z_1) = 0 \)

- General form
  - \( ax + by + cz + d = 0 \)
11.4 LINES AND PLANES IN SPACE

- Find the general equation of plane passing through $A(3, 2, 2), B(1, 5, 0)$, and $C(1, -3, 1)$
- We need to find the normal vector to the plane.
  - Find two vectors in the plane
    - $\vec{AB} = \langle 1 - 3, 5 - 2, 0 - 2 \rangle = \langle -2, 3, -2 \rangle$
    - $\vec{BC} = \langle 1 - 1, -3 - 5, 1 - 0 \rangle = \langle 0, -8, 1 \rangle$
  - Find the cross product to get a perpendicular (normal) vector
    - $\vec{n} = \vec{AB} \times \vec{BC}$
    - $\vec{n} = \begin{vmatrix}
      \hat{i} & \hat{j} & \hat{k} \\
      -2 & 3 & -2 \\
      0 & -8 & 1 
    \end{vmatrix} = \hat{i} \begin{vmatrix} 3 & -2 \\ -8 & 1 \end{vmatrix} - \hat{j} \begin{vmatrix} -2 & -2 \\ 0 & 1 \end{vmatrix} + \hat{k} \begin{vmatrix} -2 & 3 \\ 0 & -8 \end{vmatrix}$
    - $\vec{n} = 3\hat{i} + 0\hat{j} + 16\hat{k} - 0\hat{k} - 16\hat{j} - 2\hat{j}$
    - $\vec{n} = -13\hat{i} + 2\hat{j} + 16\hat{k} = \langle a, b, c \rangle$
- Fill in the general form
  - I chose $B(1, 5, 0) = \langle x_1, y_1, z_1 \rangle$
  - $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$
  - $-13(x - 1) + 2(y - 5) + 16(z - 0) = 0$
- Simplify to get general form
  - $-13x + 2y + 16z + 3 = 0$
11.4 LINES AND PLANES IN SPACE

- Angle between two planes
  - Find the angle between normal vectors
  - Normal vectors are coefficients in the equations of the plane
  - $\mathbf{n}_1 \cdot \mathbf{n}_2 = ||\mathbf{n}_1||||\mathbf{n}_2|| \cos \theta$
11.4 LINES AND PLANES IN SPACE

- Graphing planes in space
  - Find the intercepts
  - Plot the intercepts
  - Draw a triangle to represent the plane
- Sketch $3x + 4y + 6z = 24$
  - $x$-int $3x = 24 \rightarrow x = 8$
  - $y$-int $4y = 24 \rightarrow y = 6$
  - $z$-int $6z = 24 \rightarrow z = 4$
11.4 LINES AND PLANES IN SPACE

- Distance between a Point and a Plane
- \[ D = \|proj_{\mathbf{n}} \overrightarrow{PQ}\| \]
- \[ D = \left| \frac{\overrightarrow{PQ} \cdot \mathbf{n}}{\|\mathbf{n}\|} \right| \]